Operations Research Letters 40 (2012) 89–95

Contents lists available at SciVerse ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

# A stochastic program based lower bound for assemble-to-order inventory systems

Martin I. Reiman *, Qiong Wang

*Alcatel-Lucent Bell Labs, Murray Hill, NJ 07974, United States*

ARTICLE INFO

*Article history:*
Received 11 May 2011
Accepted 16 November 2011
Available online 9 December 2011

*Keywords:*
Assemble-to-order (ATO)
Inventory
Stochastic program
Optimal policy
Multi-dimensional newsvendor model

ABSTRACT

In this paper we introduce a multi-stage stochastic program that provides a lower bound on the long-run average inventory cost of a general class of assemble-to-order (ATO) inventory systems. The stochastic program also motivates a replenishment policy for these systems. Our lower bound generalizes a previous result of Doğru et al. (2010) [3] for systems with identical component replenishment lead times to those with general deterministic lead times. We provide a set of sufficient conditions under which our replenishment policy, coupled with an allocation policy, attains the lower bound (and is hence optimal). We show that these sufficient conditions hold for two examples, a single product system and a special case of the generalized W model.



## 1. Introduction

In assemble-to-order (ATO) inventory systems, intermediate and finished products can be built instantaneously, obviating the need to stock any assembly or sub-assembly. All component suppliers have ample stock, but component inventories are necessary to reduce demand backlogs caused by the lead time delay between placing and receiving replenishment orders. An ATO system is managed by a replenishment policy that decides how many parts to order, and an allocation policy that decides which product to assemble. These two decisions are reviewed either continuously or periodically over time. The objective is to minimize the long-run average sum of demand backlog and inventory holding costs. We focus on a continuous-review model, but our analysis extends to periodic-review systems as well.

Rosling [5] established that optimal management of a single-product ATO system is equivalent to that of a serial system, for which the optimal policy, found by Clark and Scarf [2], had been known for many years. However, as Song and Zipkin [6] indicate, the optimal policy for multi-product ATO systems still remains to be found. To the best of our knowledge, the only provably optimal policy is for special cases of the $W$ model, which is a two-product, three component ATO system with one component serving as a common part used by both products. Doğru et al. [3] show that for ATO systems in which all components have identical replenishment lead times, the optimal inventory cost is bounded from below by the optimal solution of a particular two-stage stochastic program. This stochastic program is a relaxed version of another stochastic program, referred to as a multi-dimensional newsvendor model (see [4,7]) and used to analyze one-period ATO systems in [6]. Doğru et al. [3] also show how, for ATO systems with identical lead times, the optimal solution of the unrelaxed stochastic program naturally gives rise to a base-stock replenishment policy. They further show that, for the $W$ model (with identical lead times) this solution leads to a priority-based allocation policy. When the two products in the $W$ model have identical inventory costs, or the optimal solution of the unrelaxed stochastic program has the 'balanced capacity' property [4], this approach is exactly optimal: its resulting cost reaches the aforementioned lower bound. It is further shown in [3], via a numerical study, that the policy suggested there performs quite well for the $W$ model over a large range of parameter values.

As Doğru et al. [3] concluded, their work is only an initial step towards the use of the stochastic program (SP) for obtaining good inventory control policies. They consider systems with identical component lead times and focus policy development and evaluation on the $W$ system. Many questions remain to be answered to attempt this approach for general cases. In particular, (i) to set a benchmark, the SP-based cost lower bound needs to be generalized to systems with non-identical lead times; (ii) to make the approach practical, an efficient procedure needs to be found to solve instances of SP problems arising from ATO systems and a mapping scheme needs to be identified to 'translate' the optimal SP solution into an implementable control policy for the original systems; and (iii) to judge the effectiveness, properties of the control policy need to be explored and performance evaluation needs to be carried out. These tasks constitute an ambitious

* Corresponding author.
*E-mail addresses:* marty@alcatel-lucent.com (M.I. Reiman), chiwang@alcatel-lucent.com (Q. Wang).

0167-6377/$ – see front matter 
doi:10.1016/j.orl.2011.12.001

research agenda. This paper takes a step forward by tackling the first task: we introduce a multi-stage SP whose optimal solution provides a lower bound on the inventory cost of a general class of ATO systems. When there are $K$ different lead times the SP has $K+1$ stages. This subsumes the lower bound presented in [3], which is based on a two-stage SP. To provide support for the effectiveness of this bound, we use the solution of the SP to define a replenishment policy for ATO systems. We then introduce a set of sufficient conditions under which this replenishment policy, coupled with some feasible allocation policy, attains the lower bound and is thus optimal. We provide two examples under which these sufficient conditions hold. The first is the single product ATO system previously considered by Rosling [5]. The second is a generalized (to more than two products) $W$ model with identical inventory costs and two different lead times where the short lead time applies to the common part.

The rest of this paper is organized as follows. After presenting the ATO model in Section 2, we develop the multi-stage stochastic program in Section 3, and also prove that its optimal solution provides a lower bound on the inventory cost of the ATO systems we consider. In Section 4 we define our replenishment policy and introduce the sufficient conditions mentioned above. We then provide the two examples and prove that they satisfy the sufficient conditions.

## 2. The assemble-to-order model

We follow the development in [3], modifying it as needed to deal with non-identical lead times. There are $m$ products and $n$ components. Components are indexed according to the ascending order of their replenishment lead times, which have $K$ distinct values

$$L_K > L_{K-1} > \cdots > L_1 > 0.$$

For notational convenience, we introduce a dummy lead time $L_0 = 0$ and let $n_0 = 0$. Thus $\{n_{k-1}+1, \ldots, n_k\}$, where $1 \le n_1 < \cdots < n_K = n$, is the index set of components of lead time $L_k$ $(k = 1, .., K)$. For $1 \le j \le n$, let $k_j$ be defined so that $L_{k_j}$ is the lead time of component $j$.

Product $i$ $(1 \le i \le m)$ uses $a_{ij}$ units of component $j$ $(1 \le j \le n)$, so the column vector $A_j = (a_{1j}, \ldots, a_{mj})'$ (where here and in the rest of the paper, apostrophe denotes transpose) specifies the usage of part $j$ $(1 \le j \le n)$ by all products. The matrix $A^k = (A_{n_{k-1}+1}, \ldots, A_{n_k})$ specifies the use of components of lead time $L_k$ $(k = 1, \ldots, K)$ by all products, and the matrix $A = (A^1, \ldots, A^K)$ specifies the use of all components.

Let $B_i(t)$ denote the backlog level of product $i$, $1 \le i \le m$, and $I_j(t)$ denote the inventory on hand of part $j$, $1 \le j \le n$. Let $\mathcal{R}_j(t)$ denote the total replenishment orders for part $j$ placed between times $-L_{k_j}$ and $t$ for $t \ge -L_{k_j}$, $1 \le j \le n$. We consider a continuous time system and take all sample paths to be right continuous. Throughout the paper $t-$ denotes the moment immediately before $t$. The initial state of the system is given by $B_i(0-)$ $(1 \le i \le m)$, $I_j(0-)$ $(1 \le j \le m)$, and $\mathcal{R}_j(t)$ for $1 \le j \le n$ and $-L_{k_j} \le t < 0$. Let

$$\mathbf{B}(t) = (B_1(t), \ldots, B_m(t))', \qquad \mathbf{I}(t) = (I_1(t), \ldots, I_n(t))', \quad t \ge 0,$$

and let

$$\mathbf{I}^k(t) = (I_{n_{k-1}+1}(t), \ldots, I_{n_k}(t))', \quad t \ge 0,$$

be inventories of all parts of lead time $L_k$, $1 \le k \le K$. Define

$$\mathcal{R}^k(t) = (\mathcal{R}_{n_{k-1}+1}(t), \ldots, \mathcal{R}_{n_k}(t))', \quad t \ge -L_k, \ 1 \le k \le K$$

and

$$\mathcal{R}(t) = (\mathcal{R}_1(t), \ldots, \mathcal{R}_n(t))', \quad t \ge 0.$$

Let $\mathcal{D}_i(t)$ denote the amount of product $i$ demand that arrives during $[0, t]$ and let

$$\mathcal{D}(t) = (\mathcal{D}_1(t), \ldots, \mathcal{D}_m(t))', \quad t \ge 0.$$

We assume that $\{\mathcal{D}(t), t \ge 0\}$ is a compound Poisson process and that $\mathbf{E}[\mathcal{D}_i(1)] < \infty$, $1 \le i \le m$. Note that we allow dependence between demands for different products. Let

$$\mathbf{D}^k(t) \equiv \mathcal{D}(t - L_{k-1}) - \mathcal{D}(t - L_k), \quad t \ge L_k,$$

be the demand that arrives during $(t - L_k, t - L_{k-1}]$, $1 \le k \le K$. Define

$$\bar{\mathbf{D}}^k(t) = \mathbf{D}^K(t) + \cdots + \mathbf{D}^{k+1}(t)$$

as the demand that arrives during $(t - L_K, t - L_k]$, $1 \le k \le K$, and

$$\bar{\mathbf{D}}(t) = \mathbf{D}^K(t) + \cdots + \mathbf{D}^1(t)$$

as the demand that arrives during $(t - L_K, t]$. Because $\mathcal{D}(t)$ is stationary, for $t \ge L_K$ the distributions of $\mathbf{D}^k(t)$, $\bar{\mathbf{D}}^k(t)$ and $\bar{\mathbf{D}}(t)$ do not depend on $t$.

Let $\mathcal{Z}_i(t)$ denote the amount of product $i$ demand served during $[0, t]$, $1 \le i \le m$, and let

$$\mathcal{Z}(t) = (\mathcal{Z}_1(t), \ldots, \mathcal{Z}_m(t))'.$$

Define

$$\mathbf{Z}^k(t) \equiv \mathcal{Z}(t - L_{k-1}) - \mathcal{Z}(t - L_k), \quad t \ge L_k,$$

and $\bar{\mathbf{Z}}^k(t) = \mathbf{Z}^K(t) + \cdots + \mathbf{Z}^{k+1}(t)$

as the amounts of demand served during $(t - L_k, t - L_{k-1}]$ and $(t - L_K, t - L_k]$, respectively, $1 \le k \le K$. Let

$$\bar{\mathbf{Z}}(t) = \mathbf{Z}^K(t) + \cdots + \mathbf{Z}^1(t)$$

be the amount of demand served during $(t - L_K, t]$. Observe that

$$\mathbf{B}(t) = \mathbf{B}(t - L_K) + \bar{\mathbf{D}}(t) - \bar{\mathbf{Z}}(t), \quad t \ge L_K. \tag{1}$$

Let $h_j$ be the unit inventory holding cost of part $j$ $(1 \le j \le n)$, $b_i$ be the unit backlog cost of product $i$ $(1 \le i \le n)$. Let

$$\mathbf{b} = (b_1, \ldots, b_m)', \qquad \mathbf{h} = (h_1, \ldots, h_n)', \quad \mathbf{c} = \mathbf{b} + A\mathbf{h},$$

and $\mathbf{h}^k = (h_{n_{k-1}+1}, \ldots, h_{n_k})' 1 \le k \le K$. The objective is to minimize the long-run average expected cost, defined as

$$C^{\gamma,p} \equiv \limsup_{T\to\infty} \frac{1}{T} \mathbf{E}\left\{ \int_{L_K}^{T} [\mathbf{b}'\mathbf{B}(t) + \mathbf{h}'\mathbf{I}(t)]dt \right\}, \tag{2}$$

where the control is exercised through a replenishment policy $\gamma$ and an allocation policy $p$. For simplicity of presentation, we start counting cost from time $L_K$ instead of 0, which has no effect when $T \to \infty$.

We restrict our attention to policies that satisfy two types of feasibility constraints, physical and informational. The physical constraints require that (i) all inventory levels be non-negative: $\mathbf{I}(t) \ge 0$ for all $t \ge 0$, and (ii) all backlog levels be non-negative: $\mathbf{B}(t) \ge 0$ for all $t \ge 0$. The informational constraint is that for all $t \ge 0$, $\mathcal{R}(t)$ and $\mathcal{Z}(t)$ be chosen using only the available information: $\mathbf{I}(0-)$, $\mathbf{B}(0-)$, $\{\mathcal{D}(s), 0 \le s \le t\}$, $\{(\mathcal{R}_j(s), -L_{k_j} \le s < t), 1 \le j \le n\}$ and $\{\mathcal{Z}(s), 0 \le s < t\}$.

### 3. Stochastic program and lower bound

Let $\bar{\mathbf{D}} = \mathbf{D}^1 + \cdots + \mathbf{D}^K$ where $\mathbf{D}^k$ $(1 \le k \le K)$ are mutually independent random vectors such that

$$\mathbf{D}^k \stackrel{d}{=} \mathbf{D}^k(t), \quad 1 \le k \le K. \tag{3}$$

Let $\mathbf{y}^k \in \mathbf{R}^+_{n_k - n_{k-1}}$ $(1 \le k \le K)$, and $\mathbf{x} \in \mathbf{R}^+_m$. We define a $K+1$ stage stochastic program as follows. Let

$$\Phi^0(\mathbf{y}^1, \dots, \mathbf{y}^K, \mathbf{x}) = -\max_{\mathbf{z} \ge 0}\{\mathbf{c}'\mathbf{z} | \mathbf{z} \le \mathbf{x}, A'\mathbf{z} \le (\mathbf{y}^1, \dots, \mathbf{y}^K)'\} \tag{4}$$

and for $k = 1, \dots, K$,

$$\Phi^k(\mathbf{y}^{k+1}, \dots, \mathbf{y}^K, \mathbf{x}) = \inf_{\mathbf{y}^k \ge 0}\left\{(\mathbf{h}^k)'\mathbf{y}^k + \mathbf{E}[\Phi^{k-1}(\mathbf{y}^k, \dots, \mathbf{y}^K, \mathbf{x} + \mathbf{D}^k)]\right\}. \tag{5}$$

For any given $\mathbf{x} \ge \mathbf{0}$ and $(\mathbf{y}^1, \dots, \mathbf{y}^K) \ge 0$ the linear program $\Phi^0$ has a finite optimal solution because $\mathbf{z} = 0$ is feasible and $\mathbf{c}'\mathbf{z}$ is bounded from above by $\mathbf{c}'\mathbf{x}$.

**Theorem 1.** *Let $(\gamma, p)$ be a feasible policy and $C^{\gamma,p}$ be the cost defined in (2). Then*

$$C^{\gamma,p} \ge \inf_{\alpha \ge 0}\left\{\mathbf{b}'(\mathbf{E}[\bar{\mathbf{D}}] + \alpha) + \Phi^K(\alpha)\right\}. \tag{6}$$

**Remark.** The lower bound in Theorem 1 arises through a relaxation of the inventory control problem. First, each point in time is myopically considered separately. Thus, focusing on a particular time $t$, only the cost rate at time $t$ is considered. Unencumbered by any prior replenishment decision (and undoing some if necessary), components with lead time $L_k$ are ordered at $t - L_k$, providing the maximum possible amount of information about demand when this decision is made. The allocation decision is relaxed in the sense that we are allowed to undo allocation decisions made during $(t - L_K, t)$ to optimize the inventory cost at time $t$. The final aspect of the relaxation involves allowing all possible values for the backlog at time $t - L_K$, and taking the one that minimizes the total expected cost.

**Proof.** We prove the theorem by showing that for all $t \ge L_K$,

$$\mathbf{E}[\mathbf{b}'\mathbf{B}(t) + \mathbf{h}'\mathbf{I}(t)] \ge \inf_{\alpha \ge 0}\left\{\mathbf{b}'(\mathbf{E}[\bar{\mathbf{D}}] + \alpha) + \Phi^K(\alpha)\right\}.$$

During the period $(t - L_K, t]$, the total amount of demand that can be served is $\mathbf{B}(t - L_K) + \bar{\mathbf{D}}(t)$. The total supply of components of lead time $L_k$ is

$$\mathbf{y}^k(t) = \mathbf{I}^k(t - L_K) + \mathcal{R}^k(t - L_k) - \mathcal{R}^k(t - L_K - L_k).$$

The total component supply minus its consumption yields the on-hand inventory at time $t$, i.e.,

$$\mathbf{I}^k(t) = \mathbf{y}^k(t) - (A^k)'\bar{\mathbf{Z}}(t). \tag{7}$$

Following (1), (7), and using $\mathbf{c}$ to replace $\mathbf{b} + A\mathbf{h}$,

$$\mathbf{b}'\mathbf{B}(t) + \mathbf{h}'\mathbf{I}(t)' = \mathbf{b}'(\mathbf{B}(t - L_K) + \bar{\mathbf{D}}(t)) + \sum_{k=1}^K (\mathbf{h}^k)'\mathbf{y}^k(t) - \mathbf{c}'\bar{\mathbf{Z}}(t) \tag{8}$$

where

$$A'\bar{\mathbf{Z}}(t) \le (\mathbf{y}^1(t), \dots, \mathbf{y}^K(t))', \qquad \bar{\mathbf{Z}}(t) \le \mathbf{B}(t - L_K) + \bar{\mathbf{D}}(t).$$

Feasibility of $\bar{\mathbf{Z}}(t)$ in the inventory system implies that $\bar{\mathbf{Z}}(t)$ is feasible for (4) with $\mathbf{y}^k = \mathbf{y}^k(t)$ $(1 \le k \le K)$ and $\mathbf{x} = \mathbf{B}(t - L_K) + \bar{\mathbf{D}}(t)$. Thus

$$-\mathbf{c}'\bar{\mathbf{Z}}(t) \ge \Phi^0(\mathbf{y}^1(t), \dots, \mathbf{y}^K(t), \mathbf{B}(t - L_K) + \bar{\mathbf{D}}(t)). \tag{9}$$

We now introduce the filtration $\{\mathcal{F}_t, t \ge 0\}$, where

$$\mathcal{F}_t = \sigma\{\mathbf{B}(0-), \mathbf{I}(0-), \mathcal{R}_j(s), -L_{k_j} \le s \le t, 1 \le j \le n; \mathcal{D}(s), \mathcal{Z}(s), 0 \le s \le t\},$$

to represent the information available after any decision at time $t$ has been made. Note that

$$\mathcal{F}_{t-L_K} \subseteq \mathcal{F}_{t-L_{K-1}} \subseteq \cdots \subseteq \mathcal{F}_t. \tag{10}$$

Using (8), the definitions of $\mathbf{y}^k(t)$ $(k = 1, 2, .., K)$, and (9), for $t \ge L_K$

$$\begin{aligned}
&\mathbf{E}[\mathbf{b}'\mathbf{B}(t) + \mathbf{h}'\mathbf{I}(t)] \\
&= \mathbf{b}'\mathbf{E}[\bar{\mathbf{D}}(t)] + \mathbf{E}\left[\mathbf{b}'\mathbf{B}(t - L_K) + \sum_{k=1}^K (\mathbf{h}^k)'\mathbf{y}^k(t) - \mathbf{c}'\bar{\mathbf{Z}}(t)\right] \\
&= \mathbf{b}'\mathbf{E}[\bar{\mathbf{D}}(t)] + \mathbf{E}\left[\mathbf{E}\left[\mathbf{b}'\mathbf{B}(t - L_K) + \sum_{k=1}^K (\mathbf{h}^k)'\mathbf{y}^k(t) - \mathbf{c}'\bar{\mathbf{Z}}(t) \middle| \mathcal{F}_{t-L_K}\right]\right] \\
&\ge \mathbf{b}'\mathbf{E}[\bar{\mathbf{D}}(t)] + \mathbf{E}\left[\mathbf{E}\left[\mathbf{b}'\mathbf{B}(t - L_K) + \sum_{k=1}^K (\mathbf{h}^k)'\mathbf{y}^k(t) + \Phi^0(\mathbf{y}^1(t), \dots, \mathbf{y}^K(t), \mathbf{B}(t - L_K) + \bar{\mathbf{D}}(t)) \middle| \mathcal{F}_{t-L_K}\right]\right].
\end{aligned} \tag{11}$$

Because $\mathbf{B}(t - L_K)$ is $\mathcal{F}_{t-L_K}$ measurable, and $\mathbf{B}(t - L_K) \ge 0$ by the required feasibility of the control policy,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{b}'\mathbf{B}(t - L_K) + \sum_{k=1}^K (\mathbf{h}^k)'\mathbf{y}^k(t) + \Phi^0(\mathbf{y}^1(t), \dots, \mathbf{y}^K(t), \mathbf{B}(t - L_K) + \bar{\mathbf{D}}(t)) \middle| \mathcal{F}_{t-L_K}\right] \\
&= \mathbf{b}'\mathbf{B}(t - L_K) + \mathbf{E}\left[\sum_{k=1}^K (\mathbf{h}^k)'\mathbf{y}^k(t) + \Phi^0(\mathbf{y}^1(t), \dots, \mathbf{y}^K(t), \mathbf{B}(t - L_K) + \bar{\mathbf{D}}(t)) \middle| \mathcal{F}_{t-L_K}\right] \\
&\ge \inf_{\alpha \ge 0}\left\{\mathbf{b}'\alpha + \mathbf{E}\left[\sum_{k=1}^K (\mathbf{h}^k)'\mathbf{y}^k(t) + \Phi^0(\mathbf{y}^1(t), \dots, \mathbf{y}^K(t), \alpha + \bar{\mathbf{D}}(t)) \middle| \mathcal{F}_{t-L_K}\right]\right\}.
\end{aligned} \tag{12}$$

Similarly, since $\mathbf{y}^K(t)$ is $\mathcal{F}_{t-L_K}$ measurable and $\mathbf{y}^K(t) \ge 0$,

$$\begin{aligned}
&\mathbf{E}\left[\sum_{k=1}^K (\mathbf{h}^k)'\mathbf{y}^k(t) + \Phi^0(\mathbf{y}^1(t), \dots, \mathbf{y}^K(t), \alpha + \bar{\mathbf{D}}(t)) \middle| \mathcal{F}_{t-L_K}\right] \\
&\ge \inf_{\mathbf{y}^K \ge 0}\left\{(\mathbf{h}^K)'\mathbf{y}^K + \mathbf{E}\left[\sum_{k=1}^{K-1} (\mathbf{h}^k)'\mathbf{y}^k(t) + \Phi^0(\mathbf{y}^1(t), \dots, \mathbf{y}^{K-1}(t), \mathbf{y}^K, \alpha + \bar{\mathbf{D}}(t)) \middle| \mathcal{F}_{t-L_K}\right]\right\} \\
&= \inf_{\mathbf{y}^K \ge 0}\left\{(\mathbf{h}^K)'\mathbf{y}^K + \mathbf{E}\left[\mathbf{E}\left[\sum_{k=1}^{K-1} (\mathbf{h}^k)'\mathbf{y}^k(t) + \Phi^0(\mathbf{y}^1(t), \dots, \mathbf{y}^{K-1}(t), \mathbf{y}^K, \alpha + \bar{\mathbf{D}}(t)) \middle| \mathcal{F}_{t-L_{K-1}}\right] \middle| \mathcal{F}_{t-L_K}\right]\right\} \\
&\ge \inf_{\mathbf{y}^K \ge 0}\left\{(\mathbf{h}^K)'\mathbf{y}^K + \mathbf{E}\left[\inf_{\mathbf{y}^{K-1} \ge 0}\left\{(\mathbf{h}^{K-1})'\mathbf{y}^{K-1}\right.\right.\right.
\end{aligned}$$

$$+\mathbf{E}\Bigg[\sum_{k=1}^{K-2}(\mathbf{h}^k)'\mathbf{y}^k(t) + \Phi^0(\mathbf{y}^1(t),\ldots,\mathbf{y}^{K-2}(t),\mathbf{y}^{K-1},\mathbf{y}^K,\alpha+\bar{\mathbf{D}}(t))\Bigg|\mathcal{F}_{t-L_{K-1}}\Bigg]\Bigg\}\Bigg|\mathcal{F}_{t-L_K}\Bigg]\Bigg\},$$

where the equality comes from (10) and Theorem 34.4 of [1]. Applying the same argument recursively, we obtain

$$\begin{aligned}
&\mathbf{E}\Bigg[\sum_{k=1}^{K}(\mathbf{h}^k)'\mathbf{y}^k(t)+\Phi^0(\mathbf{y}^1(t),\ldots,\mathbf{y}^K(t),\alpha+\bar{\mathbf{D}}(t))\Bigg|\mathcal{F}_{t-L_K}\Bigg]\\
&\geq \inf_{\mathbf{y}^K\geq 0}\Bigg\{(\mathbf{h}^K)'\mathbf{y}^K+\mathbf{E}\Bigg[\inf_{\mathbf{y}^{K-1}\geq 0}\Bigg\{(\mathbf{h}^{K-1})'\mathbf{y}^{K-1}\\
&\quad+\mathbf{E}\Bigg[\inf_{\mathbf{y}^{K-2}\geq 0}\Bigg\{(\mathbf{h}^{K-2})'\mathbf{y}^{K-2}+\cdots+\mathbf{E}\Bigg[\inf_{\mathbf{y}^1\geq 0}\Bigg\{(\mathbf{h}^1)'\mathbf{y}^1\\
&\quad+\mathbf{E}\Bigg[\Phi^0(\mathbf{y}^1,\ldots,\mathbf{y}^K,\alpha+\bar{\mathbf{D}}(t))\Bigg|\mathcal{F}_{t-L_1}\Bigg]\Bigg\}\\
&\quad\Bigg|\mathcal{F}_{t-L_2}\Bigg]\Bigg\}\Bigg|\mathcal{F}_{t-L_{K-1}}\Bigg]\Bigg\}\Bigg|\mathcal{F}_{t-L_K}\Bigg]\Bigg\}.
\end{aligned}\tag{13}$$

For instance, when $K=2$, the above becomes

$$\begin{aligned}
&\mathbf{E}\Bigg[\sum_{k=1}^{2}(\mathbf{h}^k)'\mathbf{y}^k(t)+\Phi^0(\mathbf{y}^1(t),\mathbf{y}^2(t),\alpha+\bar{\mathbf{D}}(t))\Bigg|\mathcal{F}_{t-L_2}\Bigg]\\
&\geq \inf_{\mathbf{y}^2\geq 0}\Bigg\{(\mathbf{h}^2)'\mathbf{y}^2+\mathbf{E}\Bigg[\inf_{\mathbf{y}^1\geq 0}\Bigg\{(\mathbf{h}^1)'\mathbf{y}^1\\
&\quad+\mathbf{E}\Bigg[\Phi^0(\mathbf{y}^1,\mathbf{y}^2,\alpha+\bar{\mathbf{D}}(t))\Bigg|\mathcal{F}_{t-L_1}\Bigg]\Bigg\}\Bigg|\mathcal{F}_{t-L_2}\Bigg]\Bigg\}.
\end{aligned}\tag{14}$$

Here we can write $\bar{\mathbf{D}}(t)=\mathbf{D}^1(t)+\mathbf{D}^2(t)$. By our assumption that $\{\mathcal{D}(t), t\geq 0\}$ is a compound Poisson process, with $t$ fixed, $\mathbf{D}^1(t)$ and $\mathbf{D}^2(t)$ are independent. Their distributions do not depend on $t$, and are equal to those of $\mathbf{D}^1$ and $\mathbf{D}^2$ according to (3). On the right hand side of (14), $\Phi^0(\mathbf{y}^1,\mathbf{y}^2,\alpha+\bar{\mathbf{D}}(t)) = \Phi^0(\mathbf{y}^1,\mathbf{y}^2,\alpha+\mathbf{D}^1(t)+\mathbf{D}^2(t))$. Because $\alpha+\mathbf{D}^2(t)$ is $\mathcal{F}_{t-L_1}$ measurable, and $\mathbf{D}^1(t)$ is independent of $\mathcal{F}_{t-L_1}$,

$$\begin{aligned}
&\inf_{\mathbf{y}^1\geq 0}\Bigg\{(\mathbf{h}^1)'\mathbf{y}^1+\mathbf{E}\Bigg[\Phi^0(\mathbf{y}^1,\mathbf{y}^2,\alpha+\bar{\mathbf{D}}(t))\Bigg|\mathcal{F}_{t-L_1}\Bigg]\Bigg\}\\
&=\inf_{\mathbf{y}^1\geq 0}\Bigg\{(\mathbf{h}^1)'\mathbf{y}^1+\mathbf{E}\Bigg[\Phi^0(\mathbf{y}^1,\mathbf{y}^2,\alpha+\mathbf{D}^1(t)+\mathbf{D}^2(t))\Bigg|\mathcal{F}_{t-L_1}\Bigg]\Bigg\}\\
&=\Phi^1(\mathbf{y}^2,\alpha+\mathbf{D}^2(t)),
\end{aligned}$$

where the last equality follows by (5) with $k=1$. Because $\alpha$ is $\mathcal{F}_{t-L_2}$ measurable,

$$\begin{aligned}
&\inf_{\mathbf{y}^2\geq 0}\Bigg\{(\mathbf{h}^2)'\mathbf{y}^2+\mathbf{E}\Bigg[\inf_{\mathbf{y}^1\geq 0}\Bigg\{(\mathbf{h}^1)'\mathbf{y}^1\\
&\quad+\mathbf{E}\Bigg[\Phi^0(\mathbf{y}^1,\mathbf{y}^2,\alpha+\bar{\mathbf{D}}(t))\Bigg|\mathcal{F}_{t-L_1}\Bigg]\Bigg\}\Bigg|\mathcal{F}_{t-L_2}\Bigg]\Bigg\}\\
&=\inf_{\mathbf{y}^2\geq 0}\Bigg\{(\mathbf{h}^2)'\mathbf{y}^2+\mathbf{E}\Bigg[\Phi^1(\mathbf{y}^2,\alpha+\mathbf{D}^2(t))\Bigg|\mathcal{F}_{t-L_2}\Bigg]\Bigg\}\\
&=\Phi^2(\alpha).
\end{aligned}$$

For cases where $K\geq 2$, recall that $\bar{\mathbf{D}}(t)=\mathbf{D}^1(t)+\cdots+\mathbf{D}^K(t)$, where $\mathbf{D}^1(t),\ldots,\mathbf{D}^K(t)$ are independent, and $\mathbf{D}^k(t)$ is identical in distribution to $\mathbf{D}^k$ and $\mathcal{F}_{t-L_{k-1}}$ measurable ($k=1,\ldots,K$). Applying the same procedure as in the case of $K=2$ recursively to the right-hand side of (13) and noting that $\alpha$ is $\mathcal{F}_{t-L_K}$ measurable,

$$\mathbf{E}\Bigg[\sum_{k=1}^{K}(\mathbf{h}^k)'\mathbf{y}^k(t)+\Phi^0(\mathbf{y}^1(t),\ldots,\mathbf{y}^K(t),\alpha+\bar{\mathbf{D}}(t))\Bigg|\mathcal{F}_{t-L_K}\Bigg]\geq \Phi^K(\alpha).$$

Therefore, by (11) and (12),

$$\mathbf{E}\Big[\mathbf{b}'\mathbf{B}(t)+\mathbf{h}'\mathbf{I}(t)\Big]\geq \mathbf{b}'\mathbf{E}[\bar{\mathbf{D}}(t)]+\inf_{\alpha\geq 0}\left\{\Phi^K(\alpha)+\mathbf{b}'\alpha\right\}. \quad \square$$

## 4. Replenishment policy that sometimes reaches the lower bound

Having established a lower bound on the inventory cost for ATO systems, we obviously want to know if this bound can ever be reached under a feasible inventory policy. Naturally, one would start by considering a policy that mimics the optimal solution of the SP, as we do in this section.

We define the following replenishment policy: at any time $t\geq 0$, solve $\Phi^K(\alpha)$ with $\alpha=\mathbf{B}(t)$ and denote the optimal solution by $\mathbf{y}_K^*(\tau_K)$, where $\tau_K=t+L_K$ is a reference time point that is one lead time ahead in future. Parts of such lead time ordered at time $t$ arrive at $\tau_K$.

For items of lead time $L_k$, $k=K-1,\ldots,1$: at any time $t\geq L_K-L_k$, let $\tau_k=t+L_k$, solve $\Phi^k(\mathbf{y}^{k+1},\ldots,\mathbf{y}^K,\mathbf{x})$ using

$$\mathbf{y}^{k'}=\mathbf{y}_{k'}^*(\tau_k),\quad k<k'\leq K,\quad\text{and}\quad \mathbf{x}=\mathbf{B}(\tau_k-L_K)+\bar{\mathbf{D}}^k(\tau_k).$$

Denote the optimal solution by $\mathbf{y}_k^*(\tau_k)$. Since $\mathbf{y}^{k'}$ is the optimal SP solution obtained at time $t+L_k-L_{k'}, k<k'\leq K$, and by definition, $\bar{\mathbf{D}}^k(\tau_k)$ is the demand that arrives during $(t+L_k-L_K,t]$, all information required to derive $\mathbf{y}_k^*(\tau_k)$ is available at time $t$.

Recall that in the previous section, we defined $\mathbf{y}^k(t)$ as the supply of components with lead time $L_k$ for the period $(t-L_K,t]$. Here $\mathbf{y}_k^*(\tau_k)$ is the supply of these parts for the period $(\tau_k-L_K,\tau_k]$, set by the optimal solution to the SP at $t=\tau_k-L_k$. Define $\mathbf{Y}^k(t)$ to be the inventory level (on-hand + in-transit) of these parts at $t$. With a slight abuse of our previous notation we let $\bar{\mathbf{Z}}^k(t-)$ denote the amount of demand served during $(t-L_K,t-L_k)$, so that $\mathbf{Y}^k(t-)+(A^k)'\bar{\mathbf{Z}}^k(\tau_k-)$ is the supply that has already been ordered before $t$ or consumed in the period $(\tau_k-L_K,t)$. Under our policy, the manager orders

$$\left[\mathbf{y}_k^*(\tau_k)-\mathbf{Y}^k(t-)-(A^k)'\bar{\mathbf{Z}}^k(\tau_k-)\right]^+ \tag{15}$$

to bring the supply to the desired target or keep it at the current level, whichever is larger.

In the sense that the above replenishment policy requires frequent re-solving of the stochastic program, it may not be implementable in practice. We show below in the first example that for single-product systems the replenishment policy does not, in fact, require re-solving the stochastic program. For the second example only a subproblem of the SP requires re-solving (at particular epochs—see below) and this subproblem is a variant of a newsvendor problem. We leave for future work the important step of developing replenishment (and allocation) policies based on the stochastic program solution for more general ATO systems that are implementable in practice and perform well.

For any time $\tau > L_K$, an order for parts of lead time $L_k$ placed at $\tau - L_k$ would be the last batch to arrive by $\tau$, so under our policy, the supply of these parts for the period $[\tau - L_k, \tau]$ is

$$\left[\mathbf{y}_k^*(\tau) - \mathbf{Y}^k([\tau - L_k]-) - (A^k)'\bar{\mathbf{Z}}^k(\tau-)\right]^+ + \mathbf{Y}^k([\tau - L_k]-).$$

The amount used during the same period is $A^k(\bar{\mathbf{Z}}(\tau) - \bar{\mathbf{Z}}^k(\tau-))$. The difference is on-hand inventory at $\tau$,

$$\mathbf{I}^k(\tau) = \max\left[\mathbf{y}_k^*(\tau), \mathbf{Y}^k([\tau - L_k]-) + (A^k)'\bar{\mathbf{Z}}^k(\tau-)\right] - A^k\bar{\mathbf{Z}}(\tau).$$

The backlog at $\tau$ is

$$\mathbf{B}(\tau) = \mathbf{B}(\tau - L_K) + \bar{\mathbf{D}}(\tau) - \bar{\mathbf{Z}}(\tau).$$

Thus the total inventory cost at $\tau$ is

$$C(\tau) = \sum_{k=1}^{K} \mathbf{h}^k \cdot \max\left[\mathbf{y}_k^*(\tau), \mathbf{Y}^k([\tau - L_k]-) + (A^k)'\bar{\mathbf{Z}}^k(\tau-)\right] + \mathbf{b} \cdot (\mathbf{B}(\tau - L_K) + \bar{\mathbf{D}}(\tau)) - \mathbf{c} \cdot \bar{\mathbf{Z}}(\tau). \quad (16)$$

After the above preparations, we are now ready to show that our SP lower bound is reachable in some special cases.

**Lemma 2** (*Verification Lemma*)**.** *Suppose that there exists $T < \infty$ such that for all $\tau \geq T$,*

1. *for all $1 \leq k \leq K$,*
$$\mathbf{y}_k^*(\tau) \geq \mathbf{Y}^k([\tau - L_k]-) + (A^k)'\bar{\mathbf{Z}}^k(\tau-), \quad (17)$$
2. *under a feasible allocation policy,*
$$\mathbf{c} \cdot \mathbf{E}\left[\bar{\mathbf{Z}}(\tau)\right] = \mathbf{c} \cdot \mathbf{E}\left[\mathbf{z}^*(\tau)\right], \quad (18)$$
*where $\mathbf{z}^*(\tau)$ optimizes $\Phi^0(\mathbf{y}^1, \ldots, \mathbf{y}^K, \mathbf{x})$ with*
$$\mathbf{x} = \mathbf{B}(\tau - L_K) + \bar{\mathbf{D}}(\tau) \quad \text{and} \quad \mathbf{y}^k = \mathbf{y}_k^*(\tau), \quad 1 \leq k \leq K,$$
3. *the optimal solution of the SP is independent of input $\alpha$, i.e., for all $\alpha \geq 0$,*
$$\mathbf{b} \cdot \left(\mathbf{E}[\bar{\mathbf{D}}] + \alpha\right) + \Phi^K(\alpha) = \mathbf{b} \cdot \mathbf{E}[\bar{\mathbf{D}}] + \mathbf{\Phi}^K(0). \quad (19)$$

*Then the inventory control composed of the aforementioned replenishment policy and an allocation policy yielding (18) attains the SP lower bound and thus is optimal.*

**Proof.** Because $\mathbf{D}^k(\tau) \stackrel{d}{=} \mathbf{D}^k$, and $\mathbf{y}_k^*(\tau)$ are the optimal solutions of the $k$th stage SP, $1 \leq k \leq K$, the expected total inventory cost is, from (16)–(19),

$$\begin{aligned}\mathbf{E}[C(\tau)|\mathcal{F}_{\tau - L_K}] &= \mathbf{b} \cdot \mathbf{B}(\tau - L_K) \\ &\quad + \mathbf{E}\left[\sum_{k=1}^{K} \mathbf{h}^k \cdot \mathbf{y}_k^*(\tau) + \mathbf{b} \cdot \bar{\mathbf{D}}(\tau) - \mathbf{c} \cdot \mathbf{z}^*(\tau) \middle| \mathcal{F}_{\tau - L_K}\right] \\ &= \mathbf{b} \cdot (\mathbf{E}[\bar{\mathbf{D}}] + \mathbf{B}(\tau - L_K)) + \Phi^K(\mathbf{B}(\tau - L_K)) \\ &= \inf_{\alpha \geq 0}\left\{\mathbf{b} \cdot \left(\mathbf{E}\left[\bar{\mathbf{D}}\right] + \alpha\right) + \Phi^K(\alpha)\right\}. \quad \square\end{aligned}$$

Below we show two special cases in which (17)–(19) hold.

**Example 1** (*Single-Product Systems*)**.** With only one product, $A^k$ is a vector, $1 \leq k \leq K$, so all components of lead time $L_k$ can be treated as a subassembly that is used in the product as a single unit and has the holding cost $\bar{h}_k = \mathbf{h}^k \cdot A^k$. Correspondingly, the SP optimizes a scalar $y_k$ at each of the first $k$ stages, $1 \leq k \leq K$. With only one demand process, $\mathbf{z}$ becomes a scalar $z$, as do $\alpha$ and $\mathbf{D}^k$ in the SP and relevant parameters and variables in the ATO system. The SP specializes to

$$\begin{aligned}\Phi^K(\alpha) &= \min_{y_K \geq 0}\left\{\bar{h}_K y_K + \mathbf{E}\left[\Phi^{K-1}(y_K, \alpha + D^K)\right]\right\} \\ \Phi^k(y_{k+1}, \ldots, y_K, x) &= \min_{y_k \geq 0}\left\{\bar{h}_k y_k + \mathbf{E}\left[\Phi^{k-1}(y_k, \ldots, y_K, x + D^k)\right]\right\}, \\ &\quad 1 \leq k < K, \\ \Phi^0(y_1, \ldots, y^K, x) &= -\max_{z \geq 0}\{cz | z \leq \min(x, y_1, \ldots, y_K)\}, \quad (20)\end{aligned}$$

and its optimal solution is

$$\begin{aligned}y_K^* &= \alpha + s_K^*, \\ y_k^* &= y_{k+1}^* \wedge (\alpha + \bar{D}^k + s_k^*), \quad 1 \leq k < K, \\ z^* &= y_1^* \wedge (\alpha + \bar{D}), \quad (21)\end{aligned}$$

where $s_k^*$ $(1 \leq k \leq K)$ are obtained by minimizing

$$\bar{h}_k s_k + \mathbf{E}\left[\Phi^{k-1}(s_k, +\infty, \ldots, +\infty, 0)\right]$$

over $s_k \geq 0$. Correspondingly, in the ATO system, items of lead time $L_k$, $1 \leq k \leq K$, are ordered according to (15), with $y_k^*(\tau)$ (denoting $\tau \equiv t + L_k$) given by (21) using $\alpha = B(\tau - L_K)$, $\bar{D}^k = \bar{D}^k(\tau)$, and $y_{k+1}^*(\tau)$ as inputs. To mimic $z^*$, the allocation policy serves as many demands as possible whenever there are demands to serve and components to use. It is worth pointing out that the replenishment policy here is equivalent to the known optimal policy in [5]. In both cases, a constant base-stock level, $s_k^*$, $1 \leq k \leq K$ is derived by solving a newsvendor model and the order quantities of shorter lead time items are revised down from that level to accommodate the availability of longer lead time components. We leave it to interested readers to verify our claim of exact equivalence, while focusing ourselves here on the task of showing that the outcome reaches the SP lower bound.

**Proposition 3.** *In a single-product ATO system managed by the above replenishment and allocation policies, conditions (17)–(19) are satisfied. So by Lemma 2 the policy attains the SP lower bound and thus is optimal.*

**Proof.** To verify (17), we first show that for any $\sigma < \tau$,

$$y_k^*(\tau) \geq y_k^*(\sigma) - \Delta Z_K, \quad 1 \leq k \leq K, \quad (22)$$

where $\Delta Z_K = \mathcal{Z}(\tau - L_K) - \mathcal{Z}(\sigma - L_K)$.

The condition is satisfied when $k = K$ because from (21), $y_K^*(t) = B(t - L_K) + s_K^*$ $(t = \sigma, \tau)$ and

$$B(\tau - L_K) = B(\sigma - L_K) + \Delta D_K - \Delta Z_K,$$

where $\Delta D_K = \mathcal{D}(\tau - L_K) - \mathcal{D}(\sigma - L_K)$.

From (21) and using induction, (22) holds for $k < K$ if

$$B(\tau - L_K) + \bar{D}^k(\tau) - \left(B(\sigma - L_K) + \bar{D}^k(\sigma)\right) \geq -\Delta Z_K, \quad (23)$$

which is immediate from

$$\begin{aligned}&B(\tau - L_K) - B(\sigma - L_K) = \Delta D_K - \Delta Z_K \\ &\text{and} \quad \bar{D}^k(\tau) - \bar{D}^k(\sigma) = \mathcal{D}(\tau - L_k) - \mathcal{D}(\sigma - L_k) - \Delta D_K.\end{aligned}$$

We now verify (17). For $k = K$, at any time $\tau$, since $\alpha = B(\tau - L_K)$ is used in (21) and $\bar{Z}^K(\tau-) = 0$ by definition, (15) and (21) define a base-stock replenishment policy. Thus except for possibly some initial transition period, the optimal inventory position satisfies

$$s_K^* \geq Y^K([\tau - L_K]-) - B(\tau - L_K),$$

where the right-hand side is the inventory position at $\tau - L_K$ before ordering, and (17) follows because $y_K^* = s_K^* + B(\tau - L_K)$.

To prove (17) for $k < K$, let $\sigma - L_k$ be the last time before $\tau - L_k$ when parts of lead time $L_k$ are ordered. From (15), the order brings the inventory level to $y_k^*(\sigma) - \bar{Z}^k(\sigma-)$, and any demand served at $\sigma$ brings the inventory level to $y_k^*(\sigma) - \bar{Z}^k(\sigma)$. With no order placed during $(\sigma - L_k, \tau - L_k)$, the inventory level immediately before $\tau - L_k$ satisfies

$$Y^k([\tau - L_k]-) \le y_k^*(\sigma) - \bar{Z}^k(\sigma),$$

and thus

$$\begin{aligned} Y^k([\tau - L_k]-) + \bar{Z}^k(\tau) &\le y_k^*(\sigma) + \bar{Z}^k(\tau) - \bar{Z}^k(\sigma) \\ &= y_k^*(\sigma) - \Delta Z_K. \end{aligned} \tag{24}$$

Substituting $\alpha + \bar{D}^k$ with $B(\tau - L_K) + \bar{D}^k(\tau)$ in (21),

$$y_k^*(\tau) = y_{k+1}^*(\tau) \wedge \left[s_k^* + B(\tau - L_K) + \bar{D}^k(\tau)\right].$$

Applying (22) to $y_{k+1}^*(\tau)$ in the above and using (23),

$$y_k^*(\tau) \ge y_k^*(\sigma) - \Delta Z_K.$$

Combining this with (24) yields

$$y_k^*(\tau) \ge Y^k([\tau - L_k]-) + \bar{Z}^k(\tau), \tag{25}$$

so (17) follows from (25) and $\bar{Z}^k(\tau) \ge \bar{Z}^k(\tau-)$.

To verify (18), since (25) is satisfied, at any time $\tau$, $y_k^*(\tau) - \bar{Z}^1(\tau)$ is the amount of the item of lead time $L_k$, $1 \le k \le K$, available for the period $(\tau - L_1, \tau]$. Under our replenishment policy that uses (21),

$$y_K^*(\tau) \ge y_{K-1}^*(\tau) \ge \cdots \ge y_1^*(\tau).$$

Under our allocation policy, the amount served during $(\tau - L_1, \tau]$ is

$$\left[y_1^*(\tau) - \bar{Z}^1(\tau)\right] \wedge \left[B(\tau - L_1) + D^1(\tau)\right].$$

Since

$$B(\tau - L_1) = B(\tau - L_K) + \bar{D}^1(\tau) - \bar{Z}^1(\tau),$$

the amount served during $(\tau - L_K, \tau]$ is

$$\begin{aligned} &\left[y_1^*(\tau) - \bar{Z}^1(\tau)\right] \wedge \left[B(\tau - L_1) + D^1(\tau)\right] + \bar{Z}^1(\tau) \\ &= y_1^*(\tau) \wedge \left[B(\tau - L_K) + \bar{D}(\tau)\right], \end{aligned}$$

which proves (18).

To verify (19), (21) can be transformed into

$$y_k^* = \alpha + y_{k,0}^*, \quad 1 \le k \le K, \quad \text{and} \quad z^* = \alpha + z_0^*$$

where

$$\begin{aligned} &y_{K,0}^* = s_K^*, \qquad y_{k,0}^* = y_{k+1,0}^* \wedge (\bar{D}^k + s_k^*), \quad 1 \le k < K, \\ &z_0^* = y_{1,0}^* \wedge \bar{D}. \end{aligned}$$

Applying the above to (20),

$$\Phi^K(\alpha) + b\alpha = \Phi^K(0). \quad \square$$

**Example 2** (*Generalized W Model with Symmetric Cost*). Doğru et al. [3] show that in a specialized $W$ model where all components have the same lead time and the two products have the same unit inventory cost, our SP lower bound (specialized to identical lead time cases) can be reached under a base-stock replenishment policy and a myopic allocation policy that serves as many demands as possible. Below we generalize this result to non-identical lead times and multiple products.

We consider a system of $n$ ($\ge 2$) products, where each product is assembled from a common part and a unique part. All unique parts have the same lead time, $L_2$, which is longer than the lead time of the common part, $L_1$. Following the notation in [3], the common part is identified by index 0 while all unique parts and products they go into are indexed by $i = 1, \ldots, n$. Vectors $\mathbf{h}$, $\mathbf{y}$ associated with unique parts are indexed by $u$. The assumption of the same unit inventory cost is given by

$$c_i = h_0 + h_i + b_i = \bar{c}, \quad 1 \le i \le n.$$

Let $\bar{\mathbf{c}} = (\bar{c}, \ldots, \bar{c})$ and $\mathbf{e} = (1, \ldots, 1)$. The SP (4)–(5) for this case is

$$\begin{aligned} \Phi^2(\alpha) &= \min_{\mathbf{y}_u \ge 0} \left\{\mathbf{h}_u \cdot \mathbf{y}_u + \mathbf{E}\left[\Phi^1(\mathbf{y}_u, \alpha + \mathbf{D}^2)\right]\right\}, \\ \Phi^1(\mathbf{y}_u, \mathbf{x}) &= \min_{y_0 \ge 0} \left\{h_0 y_0 + \mathbf{E}\left[\Phi^0(\mathbf{y}_u, y_0, \mathbf{x} + \mathbf{D}^1)\right]\right\}, \\ \Phi^0(\mathbf{y}_u, y_0, \mathbf{x}) &= -\max_{\mathbf{z} \ge 0} \{\bar{\mathbf{c}} \cdot \mathbf{z} | \mathbf{z} \le \mathbf{x} \wedge \mathbf{y}_u, \mathbf{e} \cdot \mathbf{z} \le y_0\}. \end{aligned} \tag{26}$$

For unique parts, the optimal solution is

$$\mathbf{y}_u^*(\alpha) = \alpha + \mathbf{s}_u^*, \tag{27}$$

where $\mathbf{s}_u^* = (s_1^*, \ldots, s_n^*)$ are positive constants. Under our replenishment policy, $\alpha = \mathbf{B}(\tau - L_2)$, so all unique parts are managed by independent base-stock policies with $s_i^*$ $(1 \le i \le n)$ being the inventory position targets.

It is easy to verify that

$$\Phi^0(\mathbf{y}_u, y_0, \mathbf{x}) = -\bar{c}\left[y_0 \wedge \sum_{i=1}^n (x_i \wedge y_i)\right],$$

so

$$\Phi^0(\mathbf{y}_u + \alpha, y_0 + \mathbf{e} \cdot \alpha, \mathbf{x} + \alpha) = -\bar{\mathbf{c}} \cdot \alpha + \Phi^0(\mathbf{y}_u, y_0, \mathbf{x}),$$

and thus

$$y_0^*(\mathbf{y}_u + \alpha, \mathbf{x} + \alpha) = \mathbf{e} \cdot \alpha + \tilde{y}_0^*(\mathbf{y}_u, \mathbf{x}) \tag{28}$$

where $\tilde{y}_0^*(\mathbf{y}_u, \mathbf{x})$ optimizes

$$\min_{\tilde{y}_0 \ge 0} \left\{h_0 \tilde{y}_0 - \bar{c}\mathbf{E}\left[\tilde{y}_0 \wedge \sum_{i=1}^n (x_i + D_i^1) \wedge y_i\right]\right\}.$$

Therefore under our replenishment policy, orders of the common part are determined by (15) using

$$y_0^*(\tau) = \mathbf{e} \cdot \mathbf{B}(\tau - L_2) + \tilde{y}_0^*(\tau), \quad \tau = t + L_1,$$

where

$$\begin{aligned} \tilde{y}_0^*(\tau) = \min\Big\{y : y \ge 0, h_0 \ge \bar{c}\Big(&\mathbf{E}\big[(y+1) \wedge g(\tau) \big| \mathcal{F}_{\tau - L_1}\big] \\ &- \mathbf{E}\big[y \wedge g(\tau) \big| \mathcal{F}_{\tau - L_1}\big]\Big)\Big\} \end{aligned} \tag{29}$$

is the optimal solution to

$$\min_{\tilde{y}_0 \ge 0} \Big\{h_0 \tilde{y}_0 - \bar{c}\mathbf{E}\big[\tilde{y}_0 \wedge g(\tau) \big| \mathcal{F}_{\tau - L_1}\big]\Big\},$$

with $g(\tau) \equiv \sum_{i=1}^n (D_i^2(\tau) + D_i^1) \wedge s_i^*$.

We need to re-solve (29) when $g(\tau)$ changes. There are two types of events that change $g(\tau)$: a new demand arrival at $t$, and a prior demand arrival 'falling off' the left end of the interval used for $\mathbf{D}^2(\tau)$, which corresponds to a demand arrival at $t + L_1 - L_2$. In response to this arrival, some unique parts are ordered under the aforementioned base-stock policy. The order will arrive at $\tau = t + L_1$, the same time when a common part ordered at time $t$ can arrive, prompting consideration of ordering the latter part at $t$. There is no need to re-solve any piece of the SP at any other time.

To prove our next proposition, it is useful to observe the following: for $\sigma < \tau$, suppose $\tilde{y}_0^*(\sigma)$ is obtained from (29) with $g(\tau)$ replaced by

$$g(\sigma) = \sum_{i=1}^{n}(D_i^2(\sigma) + D_i^1) \wedge s_i^*$$

and $\mathcal{F}_{\tau-L_1}$ replaced by $\mathcal{F}_{\sigma-L_1}$ in the optimization problem. Recall that $\Delta\mathbf{D} \equiv \mathcal{D}(\tau-L_2)-\mathcal{D}(\sigma-L_2)$ is the demand that arrives during $(\sigma - L_2, \tau - L_2]$, and $D_i^2(t)$ $(t = \sigma, \tau;\ 1 \le k \le K)$ is the demand that arrives during $(t-L_2, t-L_1]$. Both are $\mathcal{F}_{\tau-L_1}$ measurable. Thus on every sample path of $\mathbf{D}^1$,

$$g(\tau) \ge \sum_{i}^{n}(D_i^2(\sigma) - \Delta D_i + D_i^1) \wedge s_i^* \ge g(\sigma) - \mathbf{e}\cdot\Delta\mathbf{D},$$

so (29) indicates that

$$\tilde{y}_0^*(\tau) \ge \tilde{y}_0^*(\sigma) - \mathbf{e}\cdot\Delta\mathbf{D}. \qquad (30)$$

**Proposition 4.** *In the generalized W model defined above, conditions (17)–(19) are satisfied under an inventory policy that orders unique parts according to base-stock levels $s_i^*$ $(1 \le i \le n)$, determines the order quantity of the common part based on (15) and (29), and serves as many demands as possible. By Lemma 2, the policy attains the SP lower bound and thus is optimal.*

**Proof.** For the same reason given in the proof of Proposition 3, (17) holds for unique parts that are managed by independent base-stock policies. To verify that the condition also applies to the common part, let $\sigma - L_1$ be the last time the part is ordered before any time $t = \tau - L_1$. The order brings the inventory level to

$$y_0^*(\sigma) - \mathbf{e}\cdot\bar{\mathbf{Z}}^1(\sigma-) = \tilde{y}_0^*(\sigma) + \mathbf{e}\cdot\left[\mathbf{B}(\sigma - L_2) - \bar{\mathbf{Z}}^1(\sigma-)\right].$$

Recall that $\bar{\mathbf{Z}}^1(\sigma-) = \mathcal{Z}([\sigma - L_1]-) - \mathcal{Z}(\sigma - L_2)$ and that

$$\mathbf{B}(\tau - L_2) = \mathbf{B}(\sigma - L_2) + \Delta\mathbf{D} - [\mathcal{Z}(\tau - L_2) - \mathcal{Z}(\sigma - L_2)].$$

After serving demands $\mathcal{Z}([\tau-L_1]-)-\mathcal{Z}([\sigma-L_1]-)$, the inventory level immediately before $\tau - L_1$ becomes

$$\begin{aligned}
Y^0([\tau - L_1]-) &= \tilde{y}_0^*(\sigma) + \mathbf{e}\cdot\left[\mathbf{B}(\sigma - L_2) - \bar{\mathbf{Z}}^1(\sigma-)\right] \\
&\quad - \mathbf{e}\cdot[\mathcal{Z}([\tau - L_1]-) - \mathcal{Z}([\sigma - L_1]-)] \\
&= \tilde{y}_0^*(\sigma) + \mathbf{e}\cdot\mathbf{B}(\sigma - L_2) \\
&\quad + \mathbf{e}\cdot[\mathcal{Z}(\sigma - L_2) - \mathcal{Z}([\tau - L_1]-)] \\
&= \tilde{y}_0^*(\sigma) + \mathbf{e}\cdot\left[\mathbf{B}(\tau - L_2) - \Delta\mathbf{D} - \bar{\mathbf{Z}}^1(\tau-)\right] \\
&\le \tilde{y}_0^*(\tau) + \mathbf{e}\cdot\left[\mathbf{B}(\tau - L_2) - \bar{\mathbf{Z}}^1(\tau-)\right] \\
&= y_0^*(\tau) - \mathbf{e}\cdot\bar{\mathbf{Z}}^1(\tau-),
\end{aligned}$$

where the inequality above makes use of (30).

To verify (18), under any policy that does not leave demands unserved when parts are available, the amount of demands of all products served during $(\tau - L_2, \tau]$ satisfies

$$\begin{aligned}
&\mathbf{e}\cdot\bar{\mathbf{Z}}(\tau) \\
&\quad = y_0^*(\tau) \wedge \left(\sum_{i=1}^{n}(\bar{D}_i(\tau) + B_i(\tau - L_2)) \wedge (B_i(\tau - L_2) + s_i^*)\right),
\end{aligned}$$

so with $\mathbf{z}^*(\tau)$ obtained as the optimal solution of the SP using $\alpha = \mathbf{B}(\tau - L_2)$,

$$\bar{\mathbf{c}}\cdot\mathbf{E}[\bar{\mathbf{Z}}(\tau)] = \bar{\mathbf{c}}\cdot\mathbf{E}[\mathbf{z}^*(\tau)].$$

Verification of (19) is immediate from (27) and (28). □

We conclude this section by highlighting the difference in sufficient conditions for reaching the cost lower bound in the *W* system between the identical lead time case discussed in [3] and the non-identical lead time case in this example. In the former case, the optimal SP solution gives rise to a base-stock policy, which is always feasible to implement in ATO systems. Consequently, as Theorem 3.5 in [3] shows, the symmetric cost condition, which allows a simple allocation policy to achieve the same optimal outcome as the last stage SP solution, is sufficient for reaching the lower bound. Nevertheless, in cases of non-identical lead times, the SP optimal solution points to a non base-stock policy that may not be feasible to carry out in ATO systems if (17) is not satisfied. In this case, the optimal inventory level prescribed by the SP solution may fall below the existing inventory level, and hence is not attainable. Interested readers may verify that our replenishment policy does not satisfy condition (17) if we modify the example here by attaching the longer lead time $L_2$ to the common part and the shorter lead time $L_1$ to all unique parts. As a consequence, the inventory cost can be strictly higher than the lower bound even under the cost symmetric condition.

## References


[1] P. Billingsley, Probability and Measure, John Wiley and Sons, New York, 1979.
[2] A.J. Clark, H. Scarf, Optimal policies for a multiechelon inventory problem, Management Science 6 (1960) 475–490.
[3] M.K. Doğru, M.I. Reiman, Q. Wang, A stochastic programming based inventory policy for assemble-to-order systems with applications to the *W* model, Operations Research 58 (2010) 849–864.
[4] J.M. Harrison, J.A. van Mieghem, Multi-resource investment strategies: Operational hedging under demand uncertainty, European Journal of Operational Research 113 (1999) 17–29.
[5] K. Rosling, Optimal inventory policies for assembly systems under random demands, Operations Research 37 (1989) 565–579.
[6] J.-S. Song, P. Zipkin, Supply chain operations: Assemble-to-order systems, in: A.G. de Kok, S.C. Graves (Eds.), Handbooks in Operations Research and Management Science, Vol. 11: Supply Chain Management, North-Holland, 2003.
[7] J.A. van Mieghem, N. Rudi, Newsvendor networks: inventory management and capacity investment with discretionary activities, Manufacturing & Service Operations Management 4 (2002) 313–335.