Manufacturers' Assortment Planning and Pricing in a Competitive Two-Tier Supply Chain

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Abstract

We consider a two-tier supply chain in which competing manufacturers choose assortments of their products to be sold to consumers through a single wholesaler. Consumers' purchasing decisions are described by the celebrated Multinomial Logit (MNL) model, and they pay the wholesaler market prices set by the latter to maximize its profit. We formulate non-cooperative games to model assortment competitions among manufacturers, for both cases when each manufacturer's wholesale prices are given inputs to its assortment decision, and when they are optimized jointly to maximize the manufacturer's profit. We also consider situations in which the maximum number of products in each manufacture's assortment is restricted by a cardinality constraint. For each case, we develop efficient solution procedures to determine a manufacturer's optimal assortment in response to its competitors' decisions and prove the existence of a pure-strategy Nash equilibrium. We also conduct numerical studies to compare equilibrium assortments between this two-tier supply chain and the case when manufacturers sell their products directly to consumers, and discuss related profit and consumer surplus implications.

Key words: two-tier supply chain, assortment planning, pricing, non-cooperative game.

1 Introduction

We consider a two-tier supply chain in which a wholesaler serves as a single intermediary that sells products from multiple manufacturers to consumers. Each manufacturer chooses an assortment of its products to sell through the wholesaler. The wholesaler sets market prices for selling these products to consumers, whose purchasing decisions follow the celebrated Multinomial Logit (MNL) model. The wholesaler and manufacturers make decisions to maximize their own profits.

The problem setting is motivated by several common industry practices. For instance, as a remnant of the Prohibition era, breweries are often required to sell their beer products through a wholesaler, whom, under many state regulations, is given an "exclusive sales territory" in which it is the sole distributor of competing beer brands from multiple breweries (Jordan and Jaffee (1987)). The wholesaler controls prices of these products in its area and sells them through a network of competing retailers (Croxall (2021)). For instance, in the Chicago Metropolitan Area, Chicago Beverage Systems is the single wholesaler that distributes brands like Constellation Brands, MillerCoors, Heineken, Guinness, and many others (*Chicago Beverage* (2020)) to more than 4,000 retailers. Intense competitions deprive retailers of their pricing power, rendering them merely "pass through" entities that sell these products with a fixed, often negligible mark-up over the market prices set by the wholesaler. Our aforementioned two-tier supply chain provides a suitable characterization of this situation.

Similar situations can be observed in the soft-drink industry, which often features "a two-tiered market structure in which local bottlers are granted exclusive geographic territories in which to distribute products trademarked by a small oligopoly of nationwide syrup producers" (Katz (1978)). For instance, in states like Alaska, Washington, Idaho, Oregon, and Hawaii, a single bottler, The Odom Corporation distributes products from Coca-Cola, Keurig Dr Pepper, AriZona Beverages, and other manufacturers (*Odom* (2019)). A powerful retailer, such as Walmart, can also fit with the role of the wholesaler in our model, by obtaining exclusive rights from manufacturers to market their products, and selling them directly to consumers at its chosen prices.

Our model includes multiple variations. We will consider cases when manufacturers choose wholesale prices jointly with assortment decisions to maximize their profits, and when wholesale prices are given. Both cases are plausible in practice. There can be situations in which as a corporate strategy, manufacturers sell their products according to nation-wide “base prices” that are not optimized for individual regions. Manufacturers may also be locked in long-term agreements with a wholesaler that prevent them from varying prices of the same product over time and across regions to accommodate changing demand and other business conditions. In these cases, wholesale prices are exogenous inputs to assortment decisions. On the other hand, the opposite applies in situations where manufacturers have sufficient bargaining power and flexibility to make timely changes of wholesale prices that are targeted at local conditions. Studying both situations allows our model to cover a larger gap in the existing literature. Similarly, we will consider the cases in which the wholesaler can only carry a limited number of products from each manufacturer, due to the limitation of its physical or virtual shelf space, and cases in which there is no such cardinality constraint.

Corresponding to each case, we formulate an assortment competition game. For each game, we develop the optimal response of an individual manufacturer to choices made by its competitors. Our analysis yields efficient solution procedures to determine the optimal assortments in a competitive two-tier supply chain. We then proceed to prove the existence and discuss the uniqueness of Nash equilibria of these games. To the best of our knowledge, these findings make novel contributions to the current literature.

We also conduct numerical studies to compare the two-tier supply chain with the model under which manufacturers sell their products directly to consumers. We identify many cases in which some manufacturers' profits are improved in the two-tier systems. The presence of the wholesaler can help to suppress sales of weaker products, to the extent that it becomes optimal for manufacturers not to include them in their assortments. As a consequence, manufacturers with stronger products may get a higher profit than the case of direct sales, and the total profit of all manufacturers can be higher as well. Interestingly, such situations can happen only when manufacturers concede a non-trivial portion of their profit margin to the wholesaler.

Literature Review. Our work is related to studies on several types of assortment problems that involve the MNL choice model, assortment and price optimizations, cardinality constraints, two-tier supply chains, and competition.

In the context of airline revenue management, Talluri and Van Ryzin (2004) show that under the MNL model and in the absence of a cardinality constraint, it is optimal to rank products according to their revenues and apply a threshold to determine the ones to be included in the assortment. Rusmevichientong and Topaloglu (2012) show that this revenue-ordered policy also maximizes the worst-case expected revenue in the robust assortment optimization problem in which parameters of the MNL model are drawn from a compact uncertainty set. Nevertheless, the revenue-ordered policy does not apply to assortment optimization problems with cardinality constraints, and a new algorithm is developed by Rusmevichientong, Shen, and Shmoys (2010) to determine the optimal assortment with $\mathcal{O}(S^2)$ complexity, where S is the total number of products. Furthermore, Davis, Gallego, and Topaloglu (2013) develop a linear programming based solution for problems with totally unimodular constraints.

We continue this line of research by considering assortment optimization problem in the two-tier supply chain. Due to the presence of the wholesaler, manufacturers face different demand functions when making their assortment decisions. We show that the revenue-ordered policy continues to apply to cases without a cardinality constraint. For models with cardinality constraints, we also design a solution procedure with $\mathcal{O}(S^2)$. We further integrate both cases into a unified solution procedure. Moreover, these solutions are developed in the broader context of determining a manufacturer's best response in assortment competition games.

Wang (2012) shows that under the standard MNL model, when prices and assortments are jointly optimized, the seller should offer a maximum number of products that is allowed by the cardinality constraint. A later study makes extensions to include reference price effects (Wang (2018)). For the more sophisticated Nested Logit (NL) model, Li and Huh (2011) show that for any given assortment, if all products have the same price sensitivity, then their profit margin should also be the same under the optimal pricing. For problems with product-dependent price sensitivities, Gallego and Wang (2014) show that under the optimal pricing, a uniform nest-level markup applies to all nests. Our supply chain structure differs from the models in these studies and we consider assortment competition instead of optimization. Nevertheless, we show that in determining the optimal response by an individual manufacturer, equal margin still applies when all products have the same price sensitivity, and it remains optimal to offer the maximum number of products.

On assortment competition, Heese and Martínez-de Albéniz (2018) consider a two-tier supply chain in which a retailer selects at most one product from each manufacturer and sets retail prices to sell these products. Besbes and Sauré (2016) study assortment-only competition and joint pricing and assortment competition in one-tier supply chains. Our model differs from the former because we assume that the retailer (referred to as the wholesaler in our case) does not select products directly and each manufacturer has not only one but multiple products. Our work differs from the latter because we consider two-tier supply chains, which make the choice model more complicated than the MNL model when a manufacturer makes decisions.

Organization of the Paper. We define our models of two-tier supply chains and formulate related competition games in Section 2. We analyze the assortment-only competition in Section 3, and competition with both pricing and assortment decisions in Section 4. We numerically study the aforementioned assortment competition among three manufacturers with different positions in Section 5, and conclude the paper in Section 6. All proofs are relegated to the Appendix A.

2 Model

Consider a two-tier supply chain with N ($N \geq 1$) manufacturers who sell their products to consumers through a single wholesaler. Let $[N] = \{1, 2, \ldots, N\}$ and S_n be the set of all products that manufacturer n can offer ($n \in [N]$). We assume that different manufacturers do not make identical products, which is generally true in practice. For instance, the beer products of Anheuser-Busch Inbev taste different from those of Heineken. Correspondingly in the model, we assume $S_i \cap S_j = \emptyset$ for any $i \neq j$ ($i, j \in [N]$). Let $c_{n,i}$ be manufacturer n's cost of providing one unit of product i ($i \in S_n, n \in [N]$). The firm determines a product assortment A_n ($A_n \subseteq S_n$, $n \in [N]$). The wholesaler pays wholesale prices $\mathbf{p}_n^w = (p_{n,1}^w, p_{n,2}^w, \ldots, p_{n,|A_n|}^w)$ to get products in manufacturer n's assortment ($n \in [N]$), and sell them to consumers at the market prices $\mathbf{p}_n^c = (p_{n,1}^c, p_{n,2}^c, \ldots, p_{n,|A_n|}^c)$. Here superscripts w and c indicate the prices are paid by the wholesaler and consumers respectively. We consider both the case where $\mathbf{p}_n^w$ ($n \in [N]$) are given parameters and the case where they are chosen by manufacturers to maximize their respective profits.

We characterize consumer choices by the MNL model (Talluri and Van Ryzin (2006), Chapter 7.2.2). Following similar formulations in the literature (e.g., Cachon and Kök (2007)), we assume the same price sensitivity applies to all products. For any product $i \in A_n, n \in [N]$, define its **random utility** as

$$U_{n,i} := \mu_{n,i} - \alpha p_{n,i}^c + \xi_{n,i}, \tag{1}$$

where $\mu_{n,i}$ is interpreted as the "quality" of product i, $\alpha > 0$ is a price sensitivity parameter, and $\xi_{n,i}$ is independent and identically distributed (i.i.d.) random variables following Gumbel distribution with location being negative Euler's constant ($= -0.5772\ldots$) and scale being 1. Thus, for any product $i \in A_n, n \in [N]$, we have $\mathbb{E}[\xi_{n,i}] = 0$ and $\mathrm{Var}[\xi_{n,i}] = \frac{\pi^2}{6}$. Consumers may consider not purchasing anything, so we assume the random utility of the no-purchase option as

$$U_0 := \mu_0 + \xi_0,$$

where μ_0 is the deterministic utility of the no-purchase option and ξ_0 is still an i.i.d. random variable with the aforementioned Gumbel distribution.

Following Besbes and Sauré (2016), define the **attraction factor** of a product by

$$v^c_{n,i} := \exp(\mu_{n,i} - \alpha p^c_{n,i}), \quad i \in A_n,\ n \in [N]. \tag{2}$$

By choosing the product with the largest random utility, consumers' purchasing probability of product i ($i \in A_n$, $n \in [N]$) is

$$q^c_{n,i} = \frac{v^c_{n,i}}{v_0 + \sum_{m\in[N]} \sum_{j\in A_m} v^c_{m,j}}, \tag{3}$$

where $v_0 := \exp(\mu_0) > 0$ is the attraction factor of the no-purchase option. Without loss of generality, we normalize v_0 to 1, i.e., let $\mu_0 = 0$ and $U_0 = \xi_0$, so given assortments A_n ($n \in [N]$), the probability of the no-purchase option is

$$q_0 = \frac{v_0}{v_0 + \sum_{m\in[N]} \sum_{j\in A_m} v^c_{m,j}} = \frac{1}{1 + \sum_{m\in[N]} \sum_{j\in A_m} v^c_{m,j}}. \tag{4}$$

Given assortments A_n and wholesale prices $\mathbf{p}^w_n$ ($n \in [N]$), the wholesaler sets market prices $\mathbf{p}^c_n$ ($n \in [N]$) to maximize its profit, i.e.,

$$\max_{\mathbf{p}^c_n, \forall n\in[N]} \sum_{n\in[N]} \left(p^c_{n,i} - p^w_{n,i}\right) q^c_{n,i}. \tag{5}$$

This problem can be solved as follows.

Following Heese and Martínez-de Albéniz (2018), define

$$v^w_{n,i} := \exp(\mu_{n,i} - \alpha p^w_{n,i} - 1) \tag{6}$$

as the **net attraction factor** of product i ($i \in A_n$, $n \in [N]$), and

$$E_n(A_n) = \sum_{i \in A_n} v^w_{n,i} \tag{7}$$

as the **attractiveness** of the assortment A_n ($n \in [N]$). Under the constraints that $p^w_{n,i} \geq c_{n,i}$ ($i \in A_n, n \in [N]$), the value of $v^w_{n,i}$ is maximized when $p^w_{n,i} = c_{n,i}$. We denote these values by

$$\bar{v}_{n,i} := \exp(\mu_{n,i} - \alpha c_{n,i} - 1), \quad i \in A_n, n \in [N]. \tag{8}$$

Li and Huh (2011) and Heese and Martínez-de Albéniz (2018) show that in this situation, the wholesaler's optimal profit is

$$\rho^*(A_1, A_2, \ldots, A_N) = W\left(\sum_{n \in [N]} E_n(A_n)\right) \Big/ \alpha,$$

which is attained by setting the market prices at

$$p^{c*}_{n,i}(A_1, A_2, \ldots, A_N) = \rho^*(A_1, A_2, \ldots, A_N) + \frac{1}{\alpha} + p^w_{n,i}, \quad i \in A_n,\ n \in [N]. \tag{9}$$

As an implication, the optimal profit margins, $p^{c*}_{n,i} - p^w_{n,i}$ ($i \in A_n, n \in [N]$), are the same across all products. Here, $W(x)$ is the Lambert W function and defined to be the inverse of the function $f(z) = z\exp(z)$, i.e., $x = \exp(W(x))W(x)$.

When the market prices $p^{c*}_{n,i}$ ($i \in A_n, n \in [N]$) maximize the wholesaler's profit, purchasing probabilities in (3) (with $v_0 = 1$) become the following functions of the net attraction factors:

$$q^{c*}_{n,i}(A_1, A_2, \ldots, A_N) \quad = \quad \frac{v^w_{n,i}}{H\left(\sum_{m \in [N]} E_m(A_m)\right)}, \quad i \in A_n,\ n \in [N], \tag{10}$$

where

$$H(x) = \exp(W(x)) + x, \quad x \geq 0. \tag{11}$$

The net attraction factor of the no-purchase option given assortments A_n ($n \in [N]$) now is

$$v_0^w(A_1, A_2, \ldots, A_N) = \exp\left(W\left(\sum_{m\in[N]} E_m(A_m)\right)\right), \tag{12}$$

and its choice probability becomes

$$q_0^{c*}(A_1, A_2, \ldots, A_N) = \frac{v_0^w(A_1, A_2, \ldots, A_N)}{H\left(\sum_{m\in[N]} E_m(A_m)\right)}. \tag{13}$$

The following lemma summarizes the relevant properties of $H(x)$ developed in Borwein and Lindstrom (2016).

Lemma 1. *Function $H(x)$ is non-negative, strictly increasing, strictly concave, and strictly log-concave on the domain $[-1/e, +\infty)$.*

Lemma 2. *Let $x(r) := \exp(-\alpha r) + \beta$, and $\alpha, \beta > 0$. Then $H(x(r))$ is strictly log-convex in r.*

The proof of Lemma 2 is based on the particular form of $H(x(r))$. The standard convexity-preservation condition for general composite functions does not apply here. For a general composite function $f(x) = h(g(x))$, $f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$. "$f$ is convex if h is convex and nondecreasing, and g is convex" (Boyd and Vandenberghe (2004)). For the composite function $\ln H(x(r))$, we have $x(r) = \exp(-\alpha r) + \beta$ is convex and $\ln H(x)$ is increasing, but the latter is concave. We use the log-convexity of $H(x(r))$ in Lemma 2 to prove the uniqueness of the optimal margin in Section 4.1.

We now formulate different versions of Nash games of assortment competitions in a two-tier supply chain. Let C_n be the maximum number of products that the wholesaler can carry for manufacturer n ($n \in [N]$), so

$$\mathcal{A}_n = \{A_n \subseteq S_n : |A_n| \leq C_n\} \tag{14}$$

is the set of all feasible assortments that manufacture n can offer ($n \in [N]$). We refer to the conditions $|A_n| \leq C_n$ as the cardinality constraints. For given assortments and thus the attractiveness of these assortments, the profit of manufacturer n is

$$g_n(A_n, A_{-n}) = \frac{\sum_{i \in A_n} (p^w_{n,i} - c_{n,i}) v^w_{n,i}}{H\left(E_n(A_n) + E_{-n}(A_{-n})\right)}, \tag{15}$$

where for a general manufacturer n ($n \in [N]$), $A_{-n} = (A_1, A_2, \ldots, A_{n-1}, A_{n+1}, \ldots, A_N)$ is the set of all assortments from other competing manufacturers and $E_{-n}(A_{-n}) = \sum\limits_{m \in [N], m \neq n} E_m(A_m)$.

In Section 3, we consider assortment competition under given wholesale prices. Let

$$r_{n,i} = p^w_{n,i} - c_{n,i} \tag{16}$$

be manufacturer n's profit margin of product i under these prices ($i \in S_n$, $n \in [N]$). Then the assortment competition game is formulated as

$$\max_{A_n \in \mathcal{A}_n} \left\{ \frac{\sum_{i \in A_n} r_{n,i} v^w_{n,i}}{H\left(E_n(A_n) + E_{-n}(A_{-n})\right)} \right\}, \quad n \in [N]. \tag{17}$$

In Section 4, we consider the case in which wholesale prices are a part of competing strategies, and the game is thus formulated as

$$\max_{A_n \in \mathcal{A}_n,\ \mathbf{p}^w_n \in \mathcal{P}^w_n} \left\{ \frac{\sum_{i \in A_n} (p^w_{n,i} - c_{n,i}) v^w_{n,i}}{H\left(E_n(A_n) + E_{-n}(A_{-n})\right)} \right\}, \quad n \in [N], \tag{18}$$

where $\mathcal{P}_n$ is the set of feasible wholesale prices defined by the constraints $p^w_{n,i} \geq c_{n,i}, \forall i \in A_n, n \in [N]$. Under both formulations, $\mathcal{A}_n$ ($n \in [N]$) are given by (14).

We conclude the section by defining the following important concepts for subsequent discussions.

Definition 1. *Let S be a set of products. An assortment $A \subseteq S$ is a* ***profit-ordered*** *solution if*

$$r_i \geq r_j, \quad \forall i \in A \text{ and } \forall j \in S \setminus A, \tag{19}$$

where r_i is the profit margin of product i ($i \in S$) and is defined in (16).

A profit-ordered solution can always be found by sorting all products according to values of r_i ($i \in S$), and the best one can be found by comparing objective values among assortments that contain i highest-sorted products ($1 \leq i \leq |S|$). The sorting takes $O(|S| \ln |S|)$ steps and the comparison takes $|S|$ steps.

Definition 2. *Let S be a set of products. An assortment $A \subseteq S$ is a **potential-margin-ordered** solution if*

$$\bar{v}_i \geq \bar{v}_j, \quad \forall i \in A \text{ and } \forall j \in S \setminus A, \tag{20}$$

where $\bar{v}_i$ is product i's maximum net attraction factor and is defined in (8).

Similar to the profit-ordered solutions, a potential-margin-ordered solution can also be determined by sorting products by $\bar{v}_i$ ($i \in S$), and the best one can be determined by comparing the objective values among the resulting $|S|$ candidates.

Definition 3. *In an assortment A, products are priced according to the **equal-margin principle** if*

$$p_i^w = r + c_i, \quad i \in A, \tag{21}$$

where p_i^w is the wholesaler price of product i, $r \geq 0$ is a constant margin, and c_i is product i's cost ($i \in A$).

By including and dropping subscript n in variables, we have been switching back and forth between our model formulations and generic definitions. It seems less confusing if all definitions are made in our problem context.

3 Assortment-Only Competition

We study the assortment competition game in (17) by first developing the best response for a general manufacturer n ($n \in [N]$) in Section 3.1. We then prove in Section 3.2 that the game has a pure-strategy Nash equilibrium.

3.1 Best Response by Assortment Choice

We develop manufacturer n's ($n \in [N]$) best response to assortment decisions of all other manufacturers. The impact of the latter decisions on manufacturer n's profit is characterized by the combined attractiveness of all products offered by all other manufacturers $E_{-n} = \sum_{m\in[N],m\neq n} E_m(A_m)$. Given E_{-n}, let manufacturer n's best-responding assortment, the corresponding attractiveness, and the resulting profit be $A_n^*(E_{-n})$, $E_n^*(E_{-n})$ and $g_n^*(E_{-n})$, i.e.,

$$g_n^*(E_{-n}) = \max_{A_n\in\mathcal{A}_n} \left\{ g_n(A_n, E_{-n}) = \frac{\sum_{i\in A_n} r_{n,i} v_{n,i}^w}{H\left(E_n(A_n) + E_{-n}\right)} \right\}, \tag{22}$$

$A_n^*(E_{-n})$ is an optimal solution, and $E_n^*(E_{-n}) = \sum_{i\in A_n^*(E_{-n})} v_{n,i}^w$. For brevity, we will drop E_{-n} from $g_n^*(E_{-n})$, $A_n^*(E_{-n})$, $E_n^*(E_{-n})$ and $g_n(A_n, E_{-n})$, with the understanding that these quantities are defined under given E_{-n}.

Below we first consider the optimal assortment when the cardinality constraint is not tight (i.e., $C_n > |A_n^*|$). Lemma 3 shows an important impact of the profit margin on the choice, which leads to a profit-ordered solution presented in Theorem 1. We then consider the optimal assortment when the cardinality constraint is effective (i.e., $C_n = |A_n^*|$), and Lemma 4 shows that A_n^* can still be determined efficiently based on pairwise comparisons of product attraction factors and profit margins. Of course, whether the cardinality constraint is effective is generally not known a priori unless it is trivialized (i.e., $|C_n| \geq S_n$). It is possible to treat the above two situations separately and compare resulting profits to determine the optimal assortment. Nevertheless, as a more direct approach, we will present a unified solution procedure that applies to both situations.

3.1.1 Best Response When the Cardinality Constraint Is Non-binding

Lemma 3. *Let i and j be any two products in S_n where $r_{n,i} > r_{n,j}$. For any given E_{-n} and for any assortment A_n that does not contain product i or j (i.e., $i, j \notin A_n$),*

$$\text{if} \quad g_n(A_n \cup \{j\}) \geq g_n(A_n), \quad \text{then} \quad g_n(A_n \cup \{i,j\}) > g_n(A_n \cup \{j\}).$$

From the lemma, one can conclude that without any restriction on the number of products in an assortment, it is never optimal to exclude a product with a higher profit margin while including any other one with a lower margin. Theorem 1 below gives an important implication of the conclusion.

Theorem 1. *When $C_n > |A_n^*|$, A_n^* is a profit-ordered assortment defined in (19).*

It has been shown in Talluri and Van Ryzin (2004) and Rusmevichientong and Topaloglu (2012) that in a one-tier supply chain, the optimal assortment can be determined by setting a threshold on the profit margins, and including all products whose profit margins meet this threshold. Theorem 1 extends this result to the two-tier supply chain model.

3.1.2 Best Response under Effective Cardinality Constraint

When the cardinality constraint becomes effective, Theorem 1 may not hold anymore. In fact, a naïve approach of picking C_n products with the highest profit margins generally does not give an optimal assortment. As Lemma 4 shows, the selection of products should be based on a more nuanced comparison.

Lemma 4. *Consider any two products $j, k \in S_n$. Without loss of generality, assume that $v_{n,k}^w \geq v_{n,j}^w$. Under any given E_{-n},*

1. *if*

$$r_{n,k}v_{n,k}^w - r_{n,j}v_{n,j}^w \geq g_n^* H'(E_n^* + E_{-n})(v_{n,k}^w - v_{n,j}^w), \tag{23}$$

then any optimal assortment that includes product j must also include product k, i.e., if $j \in A_n^*$, then $k \in A_n^*$.

2. *on the other hand, if*

$$r_{n,k}v_{n,k}^w - r_{n,j}v_{n,j}^w \leq g_n^* H'(E_n^* + E_{-n})(v_{n,k}^w - v_{n,j}^w). \tag{24}$$

then any optimal assortment that includes product k must also include product j, i.e., if $k \in A_n^*$, then $j \in A_n^*$.

Recall that g_n^* is manufacturer n's maximum profit defined in (22), E_n^* is the attractiveness of the optimal assortment A_n^*, and the function $H(E)$ strictly increases in E. Therefore, for given E_{-n}, $g_n^* H'(E_n^* + E_{-n})$ is a positive value. Lemma 4 states that given this value, any pair of products in S_n can be ranked such that the one ranked lower cannot be in the optimal assortment unless the one ranked higher is also in it.

Specifically, let j, k be a pair of products in S_n where $v_{n,k}^w \geq v_{n,j}^w$. Let

$$s_{jk} = \frac{(r_{n,k}v_{n,k}^w - r_{n,j}v_{n,j}^w)^+}{v_{n,k}^w - v_{n,j}^w}. \tag{25}$$

Then

1. If $g_n^* H'(E_n^* + E_{-n}) < s_{jk}$, apply (23) to rank product k higher than product j.

2. If $g_n^* H'(E_n^* + E_{-n}) \geq s_{jk}$, apply (24) to rank product j higher than product k.

It is helpful to point out that Lemma 4 implies that when $g_n^* H'(E_n^* + E_{-n}) = s_{jk}$, the optimal assortment must contain either both products j and k, or none of them. For convenience, we rank j higher than product k, and this will not impact the result even though both (23) and (24) apply in this case. Moreover, when $v_{n,k}^w = v_{n,j}^w$, if $r_{n,k} > r_{n,j}$, then $s_{jk} = \infty$, and thus product k is always ranked higher than product j; if $r_{n,k} < r_{n,j}$, then $s_{jk} = 0$, and thus j is always ranked higher than k; if $r_{n,j} = r_{n,k}$, the tie can be broken arbitrary since there is no distinction between the two products.

For any three products k, j, l, if

$$r_{n,k}v^w_{n,k} - r_{n,j}v^w_{n,j} \geq g^*_n H'(E^*_n + E_{-n})(v^w_{n,k} - v^w_{n,j}),$$

$$\text{and} \quad r_{n,j}v^w_{n,j} - r_{n,l}v^w_{n,l} \geq g^*_n H'(E^*_n + E_{-n})(v^w_{n,j} - v^w_{n,l}),$$

then

$$r_{n,k}v^w_{n,k} - r_{n,l}v^w_{n,l} \geq g^*_n H'(E^*_n + E_{-n})(v^w_{n,k} - v^w_{n,l}),$$

which means the above ranking is transitive, i.e., if product k is ranked higher than product j and product j is ranked higher than product l, then product k must be ranked higher than product l. As a result, if we know the value of $g^*_n H'(E^*_n + E_{-n})$, then we could rank every pair of products in S_n and then apply this transitive property to generate a complete list that ranks all products. The top C_n products on that list would be the optimal assortment in situations when $C_n = |A^*_n|$.

However, knowing $g^*_n H'(E^*_n + E_{-n})$ usually requires knowing the optimal assortment A^*_n. To break this deadlock, we need to extend our comparisons. For each aforementioned pair of products $j, k \in S_n$, let us use s_{jk} defined in (25) to divide $[0, \infty)$ into two intervals $[0, s_{jk})$ and $[s_{jk}, \infty)$. Observe that regardless what exact value $g^*_n H'(E^*_n + E_{-n})$ is, as long as it stays within a particular interval, the ranking of products k and j will not change (for instance, k is always ranked higher than j if $g^*_n H'(E^*_n + E_{-n}) \in [0, s_{jk})$). For each pair of products, there are at most one such separating point (none if one product is always ranked higher than the other). The collection of all these points will then divide $[0, \infty)$ into a number of m intervals, where

$$m \leq |S_n|(|S_n| - 1)/2 + 1.$$

By the definition of separating points s_{jk}, when $g^*_n H'(E^*_n + E_{-n})$ is within any given interval, the ranking of any pair of two products, and thus the ranking of all products, stays the same. For each of these m ranked list, choose the top C_n products to form an assortment. The optimal assortment for cases where $C_n = |A_n^*|$ can then be attained by a direct comparison of resulting profits across these m assortments. It is easy to see the number of computations and comparisons that need to be performed is $\mathcal{O}(|S_n|^2)$.

3.1.3 Integration of Solution Procedures

The developments above present two separate procedures to determine a manufacturer's optimal response respectively for two situations: the one in which one can freely choose an assortment to maximize profit and the other in which the choice has to contain the number of products the cardinality constraint specifies. When the constraint is non-trivial (i.e., $C_n < |S_n|$), it may not be possible to tell a priori which situation applies and thus to choose the appropriate procedure. This issue can be directly addressed by running two procedures in sequence: first, use the procedure in Section 3.1.1 to determine the optimal assortment; second, keep the solution if it fits the cardinality constraint, and otherwise, switch to the alternative procedure in Section 3.1.2 to find the best assortment that meets the constraint.

Despite their difference, there is no fundamental gap between the two procedures. Both can be incorporated into a unified solution procedure that follows the same steps to find the optimal assortment regardless whether the cardinality constraint is active. In Appendix B, we present the procedure and show it can also be completed in $\mathcal{O}(|S_n|^2)$ steps.

3.2 Equilibrium of Assortment-Only Competition

We prove that the competition game in (17) has a pure-strategy Nash equilibrium, starting from the following alternative formulation of the model:

$$\begin{aligned} &\max_{E_n} \{U_n(E_n, E_{-n}) := \ln G_n(E_n) - \ln H(E_n + E_{-n})\} \\ \text{where} \quad &G_n(E_n) = \max_{A_n \in \mathcal{A}_n} \left\{ \sum_{i \in A_n} r_{n,i} v^w_{n,i} : \sum_{i \in A_n} v^w_{n,i} \le E_n \right\}, \quad n \in [N], \end{aligned} \tag{26}$$

subject to

$$\mathcal{A}_n = \{A : A \subseteq S_n \text{ and } |A| \le C_n\} \quad \text{and} \quad \min_{i \in S_n}\{v^w_{n,i}\} \le E_n \le \sum_{i \in S_n} v^w_{n,i}, \quad n \in [N]. \tag{27}$$

The above shows a seemly different model from (17). In the latter case, the manufacturer chooses the assortment directly. Here, the manufacturer first chooses an upper bound on the attractiveness of the assortment, and then find an assortment to maximize the profit under this bound. Despite this difference in formulation, the lemma below shows the two problems are equivalent in their selection of the optimal assortments.

Lemma 5. *Game (26)-(27) is equivalent to the assortment competition (17), i.e., the two problems have the same equilibrium assortments.*

Since the game (26)-(27) is equivalent to the assortment competition (17), to prove the existence of an assortment equilibrium of the game (17), we instead prove the existence of an assortment attractiveness equilibrium of the game (26)-(27).

Theorem 2. *Problem (26)-(27) is a supermodular game in which for given E_m ($m \in [N], m \neq n$) and thus for given E_{-n}, manufacturer n's payoff function $U_n(E_n, E_{-n})$ is upper-semicontinuous in its strategy E_n ($n \in [N]$). Therefore, this game, and thus the game in (17) have a pure-strategy Nash equilibrium.*

Corollary 1. *Multiple pure-strategy Nash equilibria may exist in the game (17) and thus in the game (26)-(27), Among these equilibria, one assortment equilibrium Pareto-dominates all others, and in this equilibrium, each manufacturer's assortment attractiveness is no more than the corresponding attractiveness in any other equilibrium.*

A Pareto-dominant equilibrium is the one that yields a higher profit for each manufacturer than that at any other equilibrium. Since the game (26)-(27) is a supermodular game, from Lemma 4.2.2. in Topkis (1998), we can directly get Corollary 1. For the assortment competition game (17), the following example shows that it is possible to have more than one assortment equilibrium. In this example, there are two manufacturers each of which has two products H_n and L_n ($n \in \{1, 2\}$), and no cardinality constraints. The equilibrium $(\{H_1\}, \{H_2\})$ Pareto-dominates the other equilibrium $(\{H_1, L_1\}, \{H_2, L_2\})$.

Example 1. *Parameters:*

- *price sensitivity:* $\alpha = 1$.
- *quality:* $\mu_{1,H_1} = 17, \mu_{1,L_1} = 5.5, \mu_{2,H_2} = 15, \mu_{2,L_2} = 5$.
- *costs:* $c_{1,H_1} = 3, c_{1,L_1} = 2,\ c_{2,H_2} = 5, c_{2,L_2} = 2.5$.
- *profit margins:* $r_{1,H_1} = 14,\ r_{1,L_1} = 3,\ r_{2,H_2} = 10,\ r_{2,L_2} = 2.5$.

The payoff matrix in the two-tier supply chain is (with equilibrium in bold):

	$\{H_2\}$	$\{H_2, L_2\}$
$\{H_1\}$	**2.22, 1.58**	1.76, 1.57
$\{H_1, L_1\}$	2.10, 1.11	**1.79, 1.18**

Table 1: The Payoff Matrix of Assortment Competition with Two Equilibria

4 Assortment and Pricing Competition

We now analyze the assortment and pricing competition game in (18). Similar to the last section, we start by developing the best response of a general manufacturer n ($n \in [N]$) in Section 4.1, and then prove the existence of a pure-strategy Nash equilibrium in Section 4.2.

4.1 Best Response by Pricing and Assortment Choice

Let $v_{n,i}^w(p_{n,i}^w)$ be the net attraction factor of manufacturer n's ($n \in [N]$) products under its decisions on wholesale prices $p_{n,i}^w$ ($i \in S_n$). Correspondingly, the attractiveness of a given assortment A_n is

$$E_n(\mathbf{p}_n^w, A_n) = \sum_{i \in A_n} v_{n,i}^w(p_{n,i}^w) = \sum_{i \in A_n} \exp(\mu_{n,i} - \alpha p_{n,i}^w - 1), \quad A_n \in \mathcal{A}_n. \tag{28}$$

Upon denoting E_{-n} the summation of assortment attractiveness from all manufacturers other than n, i.e., $E_{-n} = \sum_{m \in [N], m \neq n} E_m(\mathbf{p}_m^w, A_m)$, manufacturer n's profit is

$$f_n^*(E_{-n}) = \max_{\mathbf{p}_n^w \in \mathcal{P}_n,\ A_n \in \mathcal{A}_n} \left\{ f_n(\mathbf{p}_n^w, A_n, E_{-n}) = \frac{\sum_{i \in A_n} (p_{n,i}^w - c_{n,i})\ v_{n,i}^w(p_{n,i}^w)}{H\left(E_n(\mathbf{p}_n^w, A_n) + E_{-n}\right)} \right\}. \tag{29}$$

For the simplicity of notation, we drop arguments of $v_{n,i}^w(p_{n,i}^w)$ and $E_n(\mathbf{p}_n^w, A_n)$, and drop E_{-n} as an argument of $f_n^*(E_{-n})$ and $f_n(\mathbf{p}_n^w, A_n, E_{-n})$ when doing so does not cause confusion. The set of feasible assortments is again defined by (14).

Below we first characterize in Lemma 6 the optimal wholesale prices under any given assortment. We then present in Lemma 7 the optimal assortment solution under this characterization.

Lemma 6. *For any given E_{-n} and assortment $A_n \in \mathcal{A}_n$, the optimal wholesale prices must follow the equal-margin principle in Definition 3. So the optimal wholesale prices are given by (21) with $r = r_n^*$, where r_n^* is the unique optimized profit margin that applies to all products in A_n.*

Lemma 7. *For any given E_{-n} and wholesale prices that follow the equal-margin-principle, manufacturer n's optimal assortment A_n^* that maximizes (29) must be a potential-margin-ordered solution that is defined in (20) with $S = S_n$, $\bar{v}_i = \bar{v}_{n,i}$, $\forall i \in S_n$, and $C_n \wedge |S_n|$ products in the assortment.*

As is well known, the optimal pricing under the MNL model results in the same profit margin across different products (Li and Huh (2011)). Lemma 6 extends the conclusion to our case in which manufacturers face different demand functions from the MNL model due to the presence of the wholesaler. With profit margins being the same, it becomes optimal to include as many products as possible in the assortment while prioritizing the selection of those associated with the higher purchasing probabilities, which, as Lemma 7 and Corollary 1 in Heese and Martínez-de Albéniz (2018) show, means to select products with higher potential margins (see (8) for the definition). Since the latter quantities depend only on products' own characters, the optimal assortment is independent of both manufacturer n's wholesale pricing decisions and all other manufacturers' competing strategies, which is an important property. Such a solution can be found by an efficient sorting algorithm with $\mathcal{O}(|S_n|\log|S_n|)$ steps for sorting $\bar{v}_{n,i}$ $(\forall i \in S_n)$. Given the assortment, the optimal wholesale prices can be subsequently determined by a one-dimensional search of the optimal profit margin r_n^*, which is a unique optimal solution of a log-concave function (see the proof of Lemma 7 for details).

4.2 Equilibrium of Pricing and Assortment Competition

Let A_n^* be manufacturer n's optimal assortment, which by Lemma 7, is independent of the choice of the margin r_n and the value of E_{-n} $(n \in [N])$. Thus the pricing and assortment competition game in (18) reduces to a competition over profit margins r_n, which is further equivalent to the following logarithm format:

$$\begin{aligned}&\max_{r_n\geq 0}\left\{U_n(r_n, r_{-n}) := \ln r_n - \alpha r_n + \ln \bar{V}_n^* - \ln H\left(E_n(r_n)+E_{-n}(r_{-n})\right)\right\}\\ &\text{where}\quad E_n(r_n)=\exp(-\alpha r_n)\bar{V}_n^* \quad\text{and}\quad \bar{V}_n^*=\sum_{i\in A_n^*}\bar{v}_{n,i}\text{ is a constant,}\quad n\in[N].\end{aligned} \tag{30}$$

In (30), $r_{-n} = (r_1, r_2, \ldots, r_{n-1}, r_{n+1}, \ldots, r_N)$ is the vector of all profit margins without r_n and $E_{-n}(r_{-n}) = \sum_{m\in[N],m\neq n} E_m(r_m)$ is the total assortment attractiveness of all remaining manufacturers.

Referring to (30), since $r_n = 1$ is a feasible margin for manufacturer n and $E_{-n}(r_{-n}) \leq \sum_{m\in[N],m\neq n} \bar{V}_m^*$, where $\bar{V}_m^* = \sum_{i\in A_m^*} \bar{v}_{m,i}$ and A_m^* is given in Lemma 7,

$$\max_{r_n\geq 0}\{U_n(r_n, r_{-n})\} \geq -\alpha + \ln \bar{V}_n^* - \ln H\left(E_n(1) + \sum_{m\in[N],m\neq n} \bar{V}_m^*\right), \quad \forall r_{-n} \geq \mathbf{0}.$$

Since $H(x) \geq 1$ for all $x \geq 0$, under the optimal choice of r_n (denoted by r_n^*),

$$\ln r_n^* - \alpha r_n^* \geq -\alpha - \ln H\left(E_n(1) + \sum_{m\in[N],m\neq n} \bar{V}_m^*\right),$$

which implies that there exists some constant $\bar{r}_n$, independent of r_{-n}, such that $r_n^* \leq \bar{r}_n$. The same analysis applies to each of the remaining competing manufacturers. This allows us to assume without loss of generality that $r_n \in [0, \bar{r}_n]$ ($n \in [N]$), and thus the game has a non-empty and compact strategy set.

Theorem 3. *Problem (30) is a supermodular game in which manufacturer n's payoff function $U_n(r_n, r_{-n})$ is continuous in r_n for each r_{-n} ($n \in [N]$). Therefore, the problem, and thus the game in (18) have a pure-strategy Nash equilibrium.*

Corollary 2. *Multiple pure-strategy Nash equilibria may exist in the game* (30) *and thus in the game (18). Among these equilibria, one equilibrium Pareto-dominates all others, and in this equilibrium, each manufacturer's profit margin is no less than its profit margin in any other equilibrium and thus the assortment attractiveness is no more than the corresponding attractiveness in any other equilibrium.*

Since the game (30) is a supermodular game, from Lemma 4.2.2. in Topkis (1998), we can directly get Corollary 2.

5 Numerical Study

Some manufacturers sell their products directly to customers. Others conduct sales through a wholesaler, an arrangement that either is required by laws and regulations (e.g., liquor sales) or fits with the necessity and benefit of their businesses (e.g., car dealerships). We have conducted extensive numerical studies on the differences in the assortment competition without cardinality constraints between these two systems, referred to as one-tier and two-tier supply chains respectively. In this section, we present our findings through the following example.

There are three manufacturers, indexed by $n = 1, 2, 3$ and referred to as M1, M2 and M3 respectively. Each manufacturer has five products they can include in its assortment. Table 2 shows quality levels and costs of these products. We assume the price sensitivity parameter $\alpha = 1$ across all products. Prices of these products in the one-tier supply chain are also given in the table, and applying a wholesale discount to these values specifies the wholesale prices in the two-tier supply chain. We will discuss the impact of setting the discount to different levels later.

manufacturer	M1					M2					M3				
product	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
quality ($\mu_{n,i}$)	27	25	20	17	6	24	22	17	15	5	22	20	16	13	4
cost ($c_{n,i}$)	10.5	8.5	7	3	2.5	11	9	7.5	6.5	3	11.5	9.5	8	7	3.5
price ($p_{n,i}$)	33.5	28.5	24	17	6.5	33	28	23.5	16.5	6	32.5	27.5	23	16	5.5

Table 2: Parameters Used in Numerical Studies

For convenience, denote product i of manufacturer n by (n, i) ($n = 1, 2, 3, i = 1, 2, \ldots, 5$). Observe from the table that for given n, $\mu_{n,i}$, $c_{n,i}$, and $p_{n,i}$ all decrease in i, so does the profit margin

$$r_{n,i} = p_{n,i} - c_{n,i}, \quad i = 1, 2, \ldots, 5, \ n = 1, 2, 3.$$

This means it costs a manufacturer more to make a higher-quality product, to sell it at a higher price, and to fetch a higher profit margin. Across different manufacturers, the quality and prices decrease for the same i as n increases, which for instance, means that the top brand of M1 is of higher quality and more expensive than the top brand of M2's product, which in turn, dominates the top brand of M3 on these metrics. The trend is opposite in costs, which means that the top brand of M1 (M2) costs less than the top brand of M2 (M3) because of the advanced technology M1 (M2) has and the scale of production. The same applies to other products down the lists.

We also assume that in the one-tier supply chain, manufacturers may not sell their products at the prices that maximize their profits. This happens in practice when a national manufacturer does not have the knowledge nor the infrastructure to set locality-specific optimal prices. In the two-tire supply chain, the local wholesaler understands its territory well enough to charge customers optimal prices to maximize its own profit. Moreover, manufacturers may be compelled to charge the wholesaler lower prices ($p^w_{n,i}$) than the prices they would charge to customers ($p_{n,i}$) when they sell the products directly. The difference between these two prices as the percentage of $p_{n,i}$ ($n = 1, 2, 3$, $i = 1, 2, \ldots, 5$) is referred to as the wholesale discount, and denoted by d, i.e., $p^w_{n,i} = (1 - d) \cdot p_{n,i}$.

Under the above specifications, we apply the standard MNL model and the procedures developed in the previous sections to determine results of the assortment competition in one-tier and two-tier supply chains respectively. Table 3 compares manufacturers' profits between the two systems with different wholesale discount rates assumed for the case of the two-tier supply chain. Besides presenting results of assortment competitions, we also include the case in which manufacturers coordinate their assortment decisions to jointly maximize their total profit. The latter profits are shown in the last column of the table.

The table shows that in comparison with the one-tier supply chain, there are across the board and significant reductions of manufacturers' profits when the wholesale discount $d = 0$. The

		competition				coordination
		M1	M2	M3	total	max. total
one-tier supply chain		5.12	1.17	0.32	6.61	7.61
two-tier supply chain	$d = 0$	2.42	0.55	0.15	3.12	3.43
	$d = 0.05$	3.60	0.66	0.16	4.42	4.87
	$d = 0.15$	5.71	0.74	0.15	6.60	7.20
	$d = 0.25$	6.88	0.66	0.12	7.66	8.87
	$d = 0.35$	7.03	0.68	0.12	7.83	8.52

Table 3: Manufacturers' Profits under Assortment Competitions and Coordination

outcome is expected: to be profitable, the wholesaler has to charge customers higher prices than the prices it pays to the manufacturers. With $d = 0$, the latter prices are the same as those in the one-tier supply chain, so products become more expensive to customers. Manufacturers then suffer from reduced sales without any improvement of their profit margins.

What needs to be asked then is whether a higher wholesale discount can make the manufacturers better off, and the answer is yes, and quite significantly so for some of them. As is shown in the same table, when the wholesale discount rises above 0, all manufacturers' profits start to improve, to the point where at high discount rates ($d = 0.25, 0.35$), both the total profit and the profit of M1 are higher than their values in the one-tier supply chain under the setting of competition. However, the table shows that profits of M2 and M3 start to decrease when the discount rate exceeds 15%. Moreover, their profits are lower than their values in the one-tier supply chain at every discount level. This suggests the profit impact is asymmetric. Under the setting of competition, stronger manufacturers can derive more profit in the two-tier supply chain and sustain the improvement at higher wholesale discounts while weaker manufacturers will be marginalized. From the perspective of maximizing the total profit of all manufacturers, a two-tier supply chain operating under a non-zero wholesale discount rate can also be a welcome feature instead of a detriment. To compare the maximum total profit under coordination, as the table shows, the maximum total profit that can be attained had the manufacturers coordinated instead of competing in their assortment decisions is substantially higher when the wholesale discount is 0.25 (8.87) and 0.35 (8.52) than the one-tier supply chain (7.61). In contrast, the maximum total profit under coordination is lower in the two-tier supply chain when the discount rate is 0.15 or below.

It helps to understand these results by looking into prices charged to consumers, which are shown in Table 4. This table also contains the assortment equilibria represented by the appearance of a price in the two supply chain systems. The wholesaler combines assortments of all manufacturers to form a portfolio of products it sells to customers. By the basic analysis of the standard MNL model, to sell these products for the maximum profit, the wholesaler prices them to make the profit margin the same for all products (Li and Huh (2011)). The table shows prices of individual products in different cases, and the last row shows the common profit marginal set by the wholesaler for all products at different wholesale discount levels. As one would expect, a deeper discount leads to a higher margin as it becomes cheaper for the wholesaler to procure products from the manufacturers. It is also in the wholesaler's interest to share the savings with customers by reducing the prices, so the table shows that for most products, the price goes down as the discount increases. For manufacturers, the resulting increases of sales are more than enough to compensate for the discount, which explains profit increases for all of them when the discount rises above zero.

Nevertheless, the effect is uneven for different products. As one may observe in Table 4, at the same discount level, customers pay significantly less for the top brands (i.e., products 1 and 2 offered by all manufacturers) than what they pay in cases when there is no or lower discount. On the other hand, the price they pay for the lower brands (e.g., products 4 and 5) barely moves or

product		one-tier	two-tier				
			$d = 0$	$d = 0.05$	$d = 0.15$	$d = 0.25$	$d = 0.35$
M1	1	33.5	35.0	33.5	31.1	29.0	27.3
	2	28.5	30.0	28.8	26.8	25.3	24.0
	3	24.0	25.5	24.5	23.0	21.9	21.1
	4	17.0	18.5	17.9	17.0	16.7	-
M2	1	33.0	34.5	33.1	30.6	28.7	26.9
	2	28.0	29.5	28.3	26.4	24.9	23.7
	3	23.5	25.0	24.0	22.6	21.5	20.8
	4	16.5	18.0	17.4	16.6	16.3	16.2
	5	6.0	7.5	7.4	7.7	8.4	9.4
M3	1	32.5	34.0	32.6	30.2	28.3	26.6
	2	27.5	29.0	27.8	26.0	24.5	23.4
	3	23	24.5	23.6	22.1	21.2	20.4
	4	16	17.5	16.9	16.2	15.9	15.9
	5	5.5	7.0	6.9	7.3	8.0	-
wholesaler profit margin			1.5	1.7	2.6	3.9	5.5

Table 4: Prices Paid by Consumers in One-Tier and Two-Tier Supply Chains with Different Wholesale Discounts

even increases with the discount. This outcome is implied by the wholesaler's profit maximization, which requires all products to have the same profit margin, so the effect of the discount is more pronounced for products that the wholesaler needs to pay more to get from manufacturers. The discrepancy allows the manufacturer that derives its profit from high-quality top brands (M1) to gain market share from giving a discount, which explains the aforementioned divergence of profit changes among manufacturers that we observe in Table 3.

Another important outcome is the assortment decisions. In all numerical case studies that compare two supply chains, we have never encountered a case in which a manufacturer will choose a smaller assortment in a two-tier supply chain when the wholesale discount $d = 0$. In fact, the proposition below, on the special case of two manufacturers, shows that offering in the absence of the wholesale discount, manufacturers are incentivized to offer larger assortments.

In this special case, manufacturer n ($n = 1, 2$) has two products, $(n, 1)$ and $(n, 2)$, that can be included in its assortment. Like the case in Table 2, product $(n, 1)$ has higher quality and profit margin than product $(n, 2)$, and thus will always be included in the assortment (unless manufacturer n offers no product at all). Consequently, the assortment competition reduces to binary choice z_n for manufacturer n, where $z_n = 0$ or 1 corresponds to excluding or including product $(n, 2)$ respectively. The resulting profit also depends on z_{3-n}, the binary choice of the other manufacturer ($n = 1, 2$).

Proposition 1. *If $z_1 = z_2 = 0$ is an equilibrium assortment in the two-tier supply chain, then it must be an equilibrium assortment in the one-tier supply chain. On the other hand, if $z_1 = z_2 = 1$ is an equilibrium assortment in the one-tier supply chain, it must also be an equilibrium in the two-tier supply chain.*

In general, a manufacturer decides whether to include a product in its assortment based on two competing considerations: how much additional profit will be generated by selling the product to new buyers; and how much exiting profit will be lost by taking away buyers of other products, especially more valuable ones. With wholesale discount $d = 0$, products will be sold at higher prices in the two-tier supply chain to make the wholesale profitable. Consequently, there will be more non-buyers and less buyers of other products, tipping the balance towards including more products in an assortment, which is what the proposition shows for the special case and what we have observed in numerical cases.

Following the above argument, in the presence of a substantial wholesale discount, products will be sold at lower prices, reducing the number of non-buyers. Thus we should expect for larger d, manufacturers will offer smaller assortments. This is exactly what we can observe in Table 4, in the one-tier supply chain and the two-tier supply chain with wholesale discount $d \leq 25\%$, M2 and M3 include all their products in their assortments, while M1 includes all but the least valuable one (so the price of product (1,5) is missing). Nevertheless, once the discount reaches to 35%, both M1 and M3 will drop a product, (1,4) and (3,5) respectively, from their assortments. Observe from Table 2 that with $d = 35\%$,

$$(1-d)p_{1,4} - c_{1,4} = 0.65 * 17 - 3 > 0 \quad \text{and} \quad (1-d)p_{3,5} - c_{3,5} = 0.65 * 5.5 - 3.5 > 0,$$

so both products are profitable for their manufacturers, but need to be excluded to protect demands for more profitable products.

The above example shows a strong manufacturer (M1 in this case) can benefit from a two-tier supply chain in which a sufficient wholesale discount applies to all products. Our numerical study also reveals that the arrangement can also be beneficial to consumers. Table 5 shows consumer surplus in the aforementioned situations. The consumer surplus is defined as

$$\mathbb{E}\left[\max_{i \in A_n, n \in \{1,2,3\}} \max\{U_{n,i}, U_0\}\right] = \mathbb{E}\left[\max_{i \in A_n, n \in \{1,2,3\}} \max\{\mu_{n,i} - \alpha p_{n,i} + \xi_{n,i}, \xi_0\}\right]. \tag{31}$$

We use Monte Carlo simulation with sample size $10,000$ to calculate the consumer surplus (31). In comparison with the one-tier supply chain, consumer surplus decreases in the two-tier supply

one-tier	two-tier				
	$d = 0$	$d = 0.05$	$d = 0.15$	$d = 0.25$	$d = 0.35$
1.55	1.16	1.25	1.48	1.74	2.00

Table 5: Consumer Surplus in One-Tier and Two-Tier Supply Chains

chain when the discount rate $d = 0$, which is not surprising from the observation that products will be sold at higher prices. It is not difficult to imagine that these prices paid by consumers will be smaller as the wholesale discount increases, and higher consumer surplus will follow, which is the trend shown in the table.

6 Concluding Remarks

In this paper, we study the manufacturers' assortment planning as well as the joint pricing and assortment planning in a competitive two-tier supply chain, where each manufacturer makes decisions under the optimal choice probabilities induced by the wholesaler's optimal market prices. We prove that all best assortment responses are profit-ordered solutions for the uncapacitated case, and all best assortment responses can be determined by a set of candidate assortments whose cardinality is polynomial in the number of products. We generalize the results for these two cases and propose a unified solution procedure. The equal-margin principle and the potential-margin-ordered solution with the maximum allowed number of products are optimal in the pricing and assortment competition. We also show that a pure-strategy Nash equilibrium exists in both of the assortment-only competition and the joint pricing and assortment competition. As an application, we conduct a numerical study to learn the impact on the profits, assortment equilibria, and consumer surplus by the existence of a wholesaler. We observe that the impact depends on the position of a manufacturer and also on the discount levels applied to the wholesale prices in the two-tier supply chain.

There are other directions needing attention for future research. First, the product-dependent price sensitivity parameters make the wholesaler's pricing problem harder to analyze. We wonder if the product-dependent price sensitivity parameters affect manufacturers' assortment-only decision or pricing and assortment decisions. Second, a three-tier supply chain, which consists of multiple manufacturers, a single wholesaler, multiple retailers, and consumers, is of interest to be analyzed. When each retailer optimizes its market prices, and the wholesaler decides the assortments and prices given to each retailer, how will it influence the manufacturers' assortments sending to the wholesaler? Third, we use the MNL model as the consumer choice model in the competitive two-tier supply chain in this paper. It is natural to discuss the manufacturers' best responses and the competition equilibrium when the consumer behavior is governed by other choice models.

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Appendix A Proofs

Proof of Lemma 2.

Proof. Since $W'(x) = \frac{W(x)}{x[W(x)+1]}$ by Borwein and Lindstrom (2016), and $x = \exp(W(x))W(x)$ by definition, then

$$H'(x) = \frac{W(x)+2}{W(x)+1}.$$

Let $h(r) = \ln H(x(r))$, then

$$h'(r) = \frac{H'(x)}{H(x)}x'(r) = -\alpha\frac{[x(r)-\beta][W(x(r))+2]}{H(x(r))[W(x(r))+1]}.$$

We prove the lemma by showing that $g(x) = -\alpha\frac{[x-\beta][W(x)+2]}{H(x)[W(x)+1]}$ strictly decreases in x for all $x > 0$. Let $u = W(x) > 0$, i.e., $x(u) = ue^u$, then $x(u)$ increases in u, and we have $H(x(u)) = e^u + ue^u = e^u(u+1)$. Apply these equalities to $g(x)$,

$$\begin{aligned} g(x(u)) &= -\alpha(ue^u - \beta)\frac{u+2}{e^u(u+1)^2} \\ &= -\alpha\frac{u(u+2)}{(u+1)^2} + \alpha\beta\frac{u+2}{e^u(u+1)^2} \\ &= -\alpha\left(1 - \frac{1}{(u+1)^2}\right) + \alpha\beta\frac{1}{e^u(u+1)^2/(u+2)}. \end{aligned}$$

Since $(u+1)^2/(u+2)$ increases in u when $u > 0$, $g(x)$ decreases in u and thus decreases in x. $\square$

Proof of Lemma 3.

Proof. For a given assortment A_n, let

$$R = \sum_{k\in A_n} r_{n,k}v^w_{n,k}.$$

then

$$g_n(A_n) = \frac{R}{H(E(A_n)+E_{-n})}, \quad g_n(A_n\cup\{j\}) = \frac{R + r_{n,j}v^w_{n,j}}{H(E(A_n)+E_{-n}+v^w_{n,j})},$$

$$\text{and } g_n(A_n\cup\{i,j\}) = \frac{R + r_{n,i}v^w_{n,i} + r_{n,j}v^w_{n,j}}{H(E(A_n)+E_{-n}+v^w_{n,i}+v^w_{n,j})}.$$

So the lemma holds if for any $X \geq 0, x_1 > 0, x_2 > 0$ and $r_1 > r_2 \geq 0$,

$$\frac{R + r_2 x_2}{H(X + x_2)} \geq \frac{R}{H(X)} \tag{32}$$

implies that

$$\frac{R + r_1 x_1 + r_2 x_2}{H(X + x_1 + x_2)} > \frac{R + r_2 x_2}{H(X + x_2)},$$

which is true because

$$\begin{aligned}
& (R + r_1 x_1 + r_2 x_2) H(X + x_2) - (R + r_2 x_2) H(X + x_1 + x_2) \\
> \; & (R + r_2 x_1 + r_2 x_2) H(X + x_2) - (R + r_2 x_2) H(X + x_1 + x_2) \quad \text{(since } r_1 > r_2\text{)} \\
= \; & x_1 \left[r_2 H(X + x_2) - (R + r_2 x_2) \frac{H(X + x_1 + x_2) - H(X + x_2)}{x_1} \right] \\
\geq \; & x_1 \left[r_2 H(X + x_2) - (R + r_2 x_2) \frac{H(X + x_2) - H(X)}{x_2} \right] \quad \text{(by the concavity of } H(x)\text{)} \\
= \; & \frac{x_1}{x_2} H(X) H(X + x_2) \left(\frac{R + r_2 x_2}{H(X + x_2)} - \frac{R}{H(X)} \right) \\
\geq \; & 0 \quad \text{(by (32))}.
\end{aligned}$$

□

Proof of Lemma 4.

Proof. If $v^w_{n,k} > v^w_{n,j}$, we first prove point 1 by contradiction. Suppose that A^*_n contains j but not k. Let

$$A^*_n = A \cup \{j\}, \quad R = \sum_{i \in A} r_{1,i} v^w_{1,i}, \quad \text{and} \quad E_n = \sum_{i \in A} v^w_{n,i}.$$

Then

$$g^*_n = \frac{R + r_{n,j} v^w_{n,j}}{H(E^*_n + E_{-n})}, \quad \text{where} \quad E^*_n = E_n + v^w_{n,j}.$$

Consider an alternative assortment $A \cup \{k\}$, which has the profit of

$$g_n = \frac{R + r_{n,k} v^w_{n,k}}{H(E_n + v^w_{n,k} + E_{-n})}.$$

Since A_n^* is optimal, $g_n^* \geq g_n$, i.e.,

$$\frac{R + r_{n,j} v_{n,j}^w}{H(E_n^* + E_{-n})} \geq \frac{R + r_{n,k} v_{n,k}^w}{H(E_n + v_{n,k}^w + E_{-n})},$$

which can be transformed into

$$\frac{R + r_{n,j} v_{n,j}^w}{H(E_n^* + E_{-n})}[H(E_n + v_{n,k}^w + E_{-n}) - H(E_n^* + E_{-n})] \geq r_{n,k} v_{n,k}^w - r_{n,j} v_{n,j}^w.$$

Since $H(x)$ is a strictly concave function and

$$E_n^* + E_{-n} = E_n + v_{n,j}^w + E_{-n} < E_n + v_{n,k}^w + E_{-n},$$

and the above implies that

$$g_n^* H'(E_n^* + E_{-n})(v_{n,k}^w - v_{n,j}^w) > r_{n,k} v_{n,k}^w - r_{n,j} v_{n,j}^w,$$

which contradicts (23).

We can prove point 2 in a similar manner, by letting

$$g_n^* = \frac{R + r_{n,k} v_{n,k}^w}{H(E_n^* + E_{-n})}, \;\; g_n = \frac{R + r_{n,j} v_{n,j}^w}{H(E_n + v_{n,j}^w + E_{-n})},$$

where

$$E_n^* = E_n + v_{n,k}^w > E_n + v_{n,j}^w,$$

and using

$$H'(E_n^* + E_{-n})(v_{n,k}^w - v_{n,j}^w) < H(E_n^* + E_{-n}) - H(E_n + v_{n,j}^k + E_{-n}).$$

to show that $g_n^* \geq g_n$ contradicts (24).

The case with $v_{n,j}^w = v_{n,k}^w$ follows trivially from the profit function in (22). □

Proof of Lemma 5.

Proof. For any given manufacturer $n \in [N]$, let $A_1, \cdots, A_{n-1}, A_{n+1}, \cdots, A_N$ be the assortments of its competitors. Observe that in both assortment competition (17) and the game (26)-(27), n's best assortment is affected by its competitors only through their aggregated attractiveness

$$E_{-n} = \sum_{m\in[N], m\neq n} E_m(A_m).$$

Correspondingly, we denote manufacturer n's sets of best assortments in (17) and (26)-(27) by $\mathcal{A}_n(E_{-n})$ and $\mathcal{E}_n(E_{-n})$ respectively. We can then prove the lemma by showing that

$$\mathcal{A}_n(E_{-n}) = \mathcal{E}_n(E_{-n}) \tag{33}$$

for all E_{-n} and all n, which we prove next.

Let $a_n^* \in \mathcal{A}_n(E_{-n})$ and

$$E_n^* = \sum_{i\in a_n^*} v_{n,i}^w.$$

Let E_n' and a_n be any feasible solution to (26)-(27), where

$$E_n' \geq \sum_{i\in a_n} v_{n,i}^w.$$

Then

$$\frac{\sum_{i\in a_n} r_{n,i}v_{n,i}^w}{H(E_n' + E_{-n})} \leq \frac{\sum_{i\in a_n} r_{n,i}v_{n,i}^w}{H(\sum_{i\in a_n} v_{n,i}^w + E_{-n})} \leq \frac{\sum_{i\in a_n^*} r_{n,i}v_{n,i}^w}{H(E_n^* + E_{-n})},$$

where the first inequality follows from that $H(x)$ is an increasing function and the second inequality holds because $a_n^* \in \mathcal{A}_n(E_{-n})$ is manufacturer n's best response in the assortment competition (17). Thus

$$a_n^* \in \mathcal{E}_n(E_{-n}).$$

Similarly, if in the game (26)-(27), $E_n'^*$ and $e_n^* \in \mathcal{E}_n(E_{-n})$ is manufacturer n's best response, where

$$E_n'^* \geq \sum_{i\in e_n^*} v_{n,i}^w$$

then for any feasible assortment a_n in (17),

$$\frac{\sum_{i\in e_n^*} r_{n,i}v_{n,i}^w}{H(E_n'^* + E_{-n})} \geq \frac{\sum_{i\in a_n} r_{n,i}v_{n,i}^w}{H(\sum_{i\in a_n} v_{n,i}^w + E_{-n})}$$

which means that

$$e_n^* \in \mathcal{A}_n(E_{-n}).$$

The above shows that (33) holds. $\square$

Proof of Theorem 2.

Proof. To show (26)-(27) is a supermodular game: the set of feasible joint strategies $(E_1, E_2, \ldots, E_N)$ is $\prod_{n=1}^N \left[\min_{i\in S_n}\{v_{n,i}^w\}, \sum_{i\in S_n} v_{n,i}^w\right]$ which is closed, bounded, nonempty, and a sublattice of $\mathcal{R}^N$. The closeness and boundedness of the set of $(E_1, E_2, \ldots, E_N)$ imply that it is compact. Since E_n is a scaler variable, given $E_{-n} = \sum_{m\in[N],m\neq n} E_m$, the payoff function $U_n(E_n, E_{-n})$ is a supermodular function of E_n ($n \in [N]$). For any (E_n^a, E_{-n}^a) and (E_n^b, E_{-n}^b) such that $E_n^a \geq E_n^b$ and $E_{-n}^a \geq E_{-n}^b$,

$$E_n^b + E_{-n}^b \leq E_n^a + E_{-n}^b \leq E_n^a + E_{-n}^a \quad \text{and} \quad E_n^b + E_{-n}^b \leq E_n^b + E_{-n}^a \leq E_n^a + E_{-n}^a.$$

By Lemma 1, $H(x)$ is a strictly concave and increasing function, and thus a log-concave function, which means that

$$-\ln H(E_n^a + E_{-n}^b) - \ln H(E_n^b + E_{-n}^a) \leq -\ln H(E_n^b + E_{-n}^b) - \ln H(E_n^a + E_{-n}^a),$$

so $U_n(E_n, E_{-n})$ has increasing differences on (E_n, E_{-n}) for each $n \in [N]$. To show $U_n(E_n, E_{-n})$ is an upper-semicontinuous function of E_n under any given E_{-n} ($n \in [N]$), observe that $-\ln H(E_n+E_{-n})$ is a continuous and decreasing function of E_n and $\ln G_n(E_n)$ is a piecewise increasing function of E_n, where the increase can only occur at a certain number of no more than $|S_n|^2$ discontinuity points (when the optimal A_n changes).

By Theorem 4.2.1. in Topkis (1998), the above conditions are sufficient for (26)-(27) to have a pure-strategy Nash equilibrium. Let the equilibrium be $(E_1^*, E_2^*, \ldots, E_N^*)$ and the associated assortments (obtained by solving $G_n(E_n^*)$ $(n \in [N])$) be A_1^*, $A_2^*, \ldots,$ A_N^* respectively. Then for given $E_{-n}^* = \sum_{m\in[N], m\neq n} E_m^*$,

$$E_n^* = \sum_{i\in A_n^*} v_{n,i}^w \quad \text{and} \quad A_n^* \text{ maximizes } \frac{\sum_{i\in A_n} r_{n,i} v_{n,i}^w}{H(\sum_{i\in A_n} v_{n,i}^w + E_{-n}^*)}.$$

Hence, $(A_1^*, A_2^*, \ldots, A_N^*)$ is a pure-strategy Nash equilibrium of (17). $\square$

Proof of Lemma 6.

Proof. Following (28)-(29), for given E_{-n} and A_n,

$$\frac{\partial f_n(\mathbf{p}_n^w, A_n)}{\partial p_{n,i}^w} = \frac{[1-\alpha(p_{n,i}^w - c_{n,i})]v_{n,i}^w}{H(E_n + E_{-n})} + \alpha v_{n,i}^w \frac{H'(E_n + E_{-n})}{H(E_n + E_{-n})} f_n(\mathbf{p}_n^w, A_n), \quad \forall i \in A_n. \tag{34}$$

Since $0 \leq f_n(\mathbf{p}_n^w, A_n) < \infty$, and $H(x)$ is concave and thus $0 \leq H'(E_n + E_{-n})/H(E_n + E_{-n}) < \infty$ for any $E_n, E_{-n} \geq 0$, then $\frac{\partial f_n(\mathbf{p}_n^w, A_n)}{\partial p_{n,i}^w} < 0$ for any sufficiently large $p_{n,i}^w$. Also, $\frac{\partial f_n(\mathbf{p}_n^w, A_n)}{\partial p_{n,i}^w} > 0$ for all $p_{n,i}^w \leq c_{n,i}$ $(i \in A_n)$. Therefore, the optimal wholesale prices

$$p_{n,i}^{w*} = c_{n,i} + r_n^* \quad \forall i \in A_n, \tag{35}$$

where $r_n^* = \frac{1}{\alpha} + [H_n^*(A_n)]' f_n^*(A_n)$ is the optimal profit margin for all products in A_n, and with slight abuse of notation,

$$H_n^*(A_n) = H\left[\sum_{i\in A_n} v_{n,i}^w(p_{n,i}^{w*}) + E_{-n}\right] \quad \text{and} \quad f_n^*(A_n) = f_n(\mathbf{p}_n^{w*}, A_n).$$

To show that r_n^* is unique, we substitute $\mathbf{p}_n^w$ with

$$r_n := p_{n,i}^w - c_{n,i}, \quad \forall i \in S_n,$$

in the objective function (29) and prove the function is log-concave in r_n. Let

$$x(r_n) = \sum_{i\in A_n} v_{n,i}^w = \exp(-\alpha r_n) \sum_{i\in A_n} \bar{v}_{n,i} > 0, \quad \forall r_n > 0,$$

where the second equality comes from (28). Then, for any given A_n, with slight abuse of the symbol, let $f_n(r_n, A_n)$ be the profit under r_n and A_n,

$$\ln f_n(r_n, A_n) = l(r_n) = \ln r_n - \alpha r_n + \ln\left(\sum_{i \in A_n} \bar{v}_{n,i}\right) - \ln\left(H(x(r_n) + E_{-n})\right) \tag{36}$$

By Lemma 2, $l(r_n)$ is concave. □

Proof of Lemma 7.

Proof. Let r_n be the common margin of all products in S_n. Then by (6), (8), and (21),

$$v^w_{n,i} = \bar{v}_{n,i} \exp(-\alpha r_n), \quad i \in S_n.$$

For an assortment $A_n \in \mathcal{A}_n$, let

$$\bar{V}(A_n) = \sum_{i \in A_n} \bar{v}_{n,i},$$

and again let $f_n(r_n, A_n)$ be the profit under r_n and A_n. Then by (29),

$$f_n(r_n, A_n) = r_n \frac{e^{-\alpha r_n} \bar{V}(A_n)}{H(e^{-\alpha r_n} \bar{V}(A_n) + E_{-n})}. \tag{37}$$

By Lemma 1, for any $x_2 > x_1 > 0$,

$$\frac{H(x_2) - H(x_1)}{x_2 - x_1} < \frac{H(x_1) - H(-\frac{1}{e})}{x_1 + \frac{1}{e}} = \frac{H(x_1)}{x_1 + \frac{1}{e}} < \frac{H(x_1)}{x_1},$$

which means that

$$\frac{x_2}{H(x_2)} > \frac{x_1}{H(x_1)}.$$

Therefore, to maximize $f_n(r_n, A_n)$ in (37), A_n must be a feasible assortment that maximizes $\bar{V}(A_n)$, which means it must contain $|S_n| \wedge C_n$ products with the largest $\bar{v}_{n,i}$ ($i \in S_n$). □

Proof of Theorem 3.

Proof. The set of feasible joint strategies $(r_1, r_2, \dots, r_N)$ is $\prod_{n=1}^{N}[0, \bar{r}_n]$ which is nonempty, compact, a sublattice of $\mathcal{R}^N$. Because r_n is a scalar, for any $r_{-n} = (r_1, r_2, r_{n-1}, r_{n+1}, \dots, r_N)$, the payoff function $U_n(r_n, r_{-n})$ is supermodular in r_n on $[0, \bar{r}_n]$ $(n \in [N])$. Moreover, $U_n(r_n, r_{-n})$ is supermodular in (r_n, r_{-n}) because any cross partial is always positive, i.e., when $E_{-n}(r_{-n}) = \sum_{m\in[N], m\neq n} E_m(r_m)$,

$$\frac{\partial^2 U_n(r_n, r_{-n})}{\partial r_i \partial r_j} = -\frac{\partial^2 [\ln H\left(E_n(r_n) + E_{-n}(r_{-n})\right)]}{\partial r_i \partial r_j}, \quad \forall i, j \in [N], i \neq j, \forall n \in [N]$$

is always positive by Lemma 2. By Theorem 2.2.2. in Simchi-Levi, Chen, and Bramel (2013), the supermodularity of $U_n(r_n, r_{-n})$ implies that it has increasing differences in (r_n, r_{-n}) for any $n \in [N]$. $U_n(r_n, r_{-n})$ is continuous in r_n for each r_{-n} $(n \in [N])$, thus the game (30) is a supermodular game. Again by Theorem 4.2.1. in Topkis (1998), the game defined in (30) has a pure-strategy Nash equilibrium denoted by $(r_1^*, r_2^*, \dots, r_N^*)$. Together with $(A_1^*, A_2^*, \dots, A_N^*)$, we conclude that $((r_1^*, A_1^*), (r_2^*, A_2^*), \dots, (r_N^*, A_N^*))$ is a pure-strategy Nash equilibrium of the game (18). □

Proof of Proposition 1.

Proof. In the one-tier supply chain, denote the profit margin and attractiveness of product (n, i) by $r_{n,i}$ and $v_{n,i}$ respectively $(n = 1, 2,\ i = 1, 2)$. The assortment competition specializes to

$$\begin{aligned} &\max_{z_n\in\{0,1\}} \left\{ g_n^I(z_n, z_{3-n}) \right\}, \\ &\text{where} \quad g_n^I(z_n, z_{3-n}) = \frac{r_{n,1}v_{n,1} + z_n r_{n,2} v_{n,2}}{1 + v_{n,1} + v_{3-n,1} + v_{n,2} z_n + z_{3-n} v_{3-n,2}}, \quad n = 1, 2. \end{aligned} \tag{38}$$

For $z_1 = z_2 = 0$ to be an equilibrium,

$$g_n^I(0, 0) \geq g_n^I(1, 0), \quad n = 1, 2,$$

which holds if and only if

$$\frac{r_{n,1}v_{n,1}}{1 + v_{n,1} + v_{3-n,1}} \geq \frac{r_{n,1}v_{n,1} + r_{n,2}v_{n,2}}{1 + v_{n,1} + v_{3-n,1} + v_{n,2}}, \quad n = 1, 2,$$

i.e.,

$$\begin{aligned}&\frac{r_{n,1}}{r_{n,2}} \geq X_n^I, \\ \text{where} \quad & X_n^I = \frac{1+v_{1,1}+v_{2,1}}{v_{n,1}}, \quad n=1,2.\end{aligned} \tag{39}$$

Similarly, for $z_1 = z_2 = 1$ to be an equilibrium,

$$g_n^I(1,1) \geq g_n^I(0,1), \quad n=1,2,$$

which holds if and only if

$$\frac{r_{n,1}v_{n,1}+r_{n,2}v_{n,2}}{1+v_{n,1}+v_{n,2}+v_{3-n,1}+v_{3-n,2}} \geq \frac{r_{n,1}v_{n,1}}{1+v_{n,1}+v_{3-n,1}+v_{3-n,2}}, \quad n=1,2,$$

i.e.,

$$\begin{aligned}&\frac{r_{n,1}}{r_{n,2}} \leq Y_n^I, \\ \text{where} \quad & Y_n^I = \frac{1+v_{1,1}+v_{2,1}+v_{3-n,2}}{v_{n,1}}, \quad n=1,2.\end{aligned} \tag{40}$$

In the two-tier supply chain, with $d=0$, the margin of product (n,i), $p_{n,i}^w - c_{n,i}$, remains to be $r_{n,i}$, and by (2) and (6), the attractiveness of the product is

$$v_{n,i}^w = v_{n,i}/e, \quad n=1,2, i=1,2.$$

Referring to (11) for the definition of $H(x)$ and denote

$$\phi(x) = eH(x/e) = \exp\left(W(x/e)+1\right) + x.$$

Then with attractiveness defined in (7) and values of $p_{n,i}^w$ and $v_{n,i}^w$ specified above, the assortment competition model (17) specializes to

$$\begin{aligned}&\max_{z_n\in\{0,1\}} \left\{g_n^{II}(z_n, z_{3-n})\right\}, \\ \text{where} \quad & g_n^{II}(z_n, z_{3-n}) = \frac{r_{n,1}v_{n,1}+z_n r_{n,2}v_{n,2}}{\phi\left(v_{1,1}+v_{2,1}+z_n v_{n,2}+z_{3-n}v_{3-n,2}\right)}, \quad n=1,2.\end{aligned} \tag{41}$$

By Lemma 1, $\phi(x)$ is non-negative, strictly increasing, and strictly concave function in x for $x \geq -1$. It is also easy to verify that $\phi(-1) = 0$. Hence for any $x \geq 0$ and $\Delta x > 0$,

$$\frac{\phi(x)}{1+x} = \frac{\phi(x) - \phi(-1)}{1+x} > \frac{\phi(x+\Delta x) - \phi(x)}{\Delta x},$$

which can be transformed into

$$\frac{\phi(x)\Delta x}{\phi(x+\Delta x) - \phi(x)} > 1 + x, \tag{42}$$

a property that we will use next.

Similar to the one-tier supply chain, $z_1 = z_2 = 0$ is an equilibrium of the assortment competition in the two-tier supply chain if and only if

$$g_n^{II}(0,0) \geq g_n^{II}(1,0), \quad n = 1, 2,$$

which can be simplify to

$$\begin{aligned} &\frac{r_{n,1}}{r_{n,2}} \geq X_n^{II}, \\ \text{where} \quad &X_n^{II} = \frac{\phi(v_{1,1} + v_{2,1})v_{n,2}}{v_{n,1}\left[\phi(v_{1,1} + v_{2,1} + v_{n,2}) - \phi(v_{1,1} + v_{2,1})\right]}, \quad n = 1, 2, \end{aligned} \tag{43}$$

and $z_1 = z_2 = 1$ is an equilibrium if and only if

$$g_n^{II}(1,1) \geq g_n^{II}(0,1), \quad n = 1, 2,$$

which can be simplify to

$$\begin{aligned} &\frac{r_{n,1}}{r_{n,2}} \leq Y_n^{II}, \\ \text{where} \quad &Y_n^{II} = \frac{\phi(v_{1,1} + v_{2,1} + v_{3-n,2})v_{n,2}}{v_{n,1}\left[\phi(v_{1,1} + v_{2,1} + v_{n,2} + v_{3-n,2}) - \phi(v_{1,1} + v_{2,1} + v_{3-n,2})\right]}, \quad n = 1, 2. \end{aligned} \tag{44}$$

Thus to prove the proposition, we only need to show that

$$X_n^I \leq X_n^{II} \quad \text{and} \quad Y_n^I \leq Y_n^{II}, \quad n = 1, 2.$$

which, according to (39)-(40) and (43)-(44), can be expanded and simplify to

$$1+v_{1,1}+v_{2,1} \le \frac{\phi(v_{1,1}+v_{2,1})v_{n,2}}{\phi(v_{1,1}+v_{2,1}+v_{n,2})-\phi(v_{1,1}+v_{2,1})}, \quad \text{and}$$

$$1+v_{1,1}+v_{2,1}+v_{3-n,2} \le \frac{\phi(v_{1,1}+v_{2,1}+v_{3-n,2})v_{n,2}}{\phi(v_{1,1}+v_{2,1}+v_{n,2}+v_{3-n,2})-\phi(v_{1,1}+v_{2,1}+v_{3-n,2})}, \quad n=1,2.$$

Both follow from (42): to prove the first inequality, let

$$\Delta x = v_{n,2} \quad \text{and} \quad x = v_{1,1}+v_{2,1}, \quad n=1,2,$$

and to prove the second inequality, let

$$\Delta x = v_{n,2} \quad \text{and} \quad x = v_{1,1}+v_{2,1}+v_{3-n,2}, \quad n=1,2.$$

□

Appendix B A Unified Approach

As is referred to in Section 3.1.3, we now integrate algorithms in Sections 3.1.1 and 3.1.2 into a unified procedure to solve both cases where $C_n > |A_n^*|$ and $C_n = |A_n^*|$. Let d be a dummy product with $v_d^w = 0$ and $r_d = 0$. We can then rank any product $k \in S_n$ and product d by applying Lemma 4 with product j replaced by d. Conditions (23) and (24) reduce to

$$r_{n,k} \ge g_n^* H'(E_n^* + E_{-n}) \quad \text{and} \quad r_{n,k} \le g_n^* H'(E_n^* + E_{-n})$$

respectively, and (25) reduces to $s_{dk} = r_{n,k}$. Product k ranks higher than d if $g_n^* H'(E_n^* + E_{-n}) \in [0, r_{n,k})$ and ranked lower if $g_n^* H'(E_n^* + E_{-n}) \in [r_{n,k}, \infty)$.

When $C_n > |A_n^*|$, adding the dummy product d to the optimal assortment A_n^* is feasible but has no impact on manufacturer n's profit. Thus one may expect that if product k is ranked higher than product d, then it should be included in A_n^* provided that doing so is allowed by the cardinality constraint. On the other hand, if k is ranked lower than d, then it should be excluded from A_n^*

even if having it in the assortment does not violate the cardinality constraint. Theorem 4 makes this formal.

Theorem 4. *For given E_{-n}, let A_n^* be manufacturer n's best-responding assortment. Then for any product $k \in A_n^*$,*

$$r_{n,k}v_{n,k}^w - r_{n,j}v_{n,j}^w \geq g_n^* H'(E_n^* + E_{-n})(v_{n,k}^w - v_{n,j}^w) \quad \textit{for all } j \notin A_n^*. \tag{45}$$

Moreover, upon the satisfaction of the above condition and the cardinality constraint, product k must be included in A_n^* if and only if

$$r_{n,k} \geq g_n^* H'(E_n^* + E_{-n}). \tag{46}$$

Proof. Inequality (45) follows directly from Lemma 4.

Let

$$g_n^* = \frac{R^*}{H(V^*)} \quad \text{where} \quad R^* = \sum_{i \in A_n^*} r_{n,i}v_{n,i}^w \quad \text{and} \quad V^* = E_n^* + E_{-n}.$$

To prove (46) must hold if $k \in A_n^*$, by the optimality of A_n^*,

$$\frac{R^*}{H(V^*)} - \frac{R^* - r_{n,k}v_{n,k}^w}{H(V^* - v_{n,k}^w)} = \frac{v_{n,k}^w}{H(V^* - v_{n,k}^w)} \left(r_{n,k} - \frac{R^*}{H(V^*)} \frac{H(V^*) - H(V^* - v_{n,k}^w)}{v_{n,k}^w} \right) > 0.$$

It follows that (46) must hold in this case because by the concavity of $H(x)$,

$$\frac{H(V^*) - H(V^* - v_{n,k}^w)}{v_{n,k}^w} > H'(V^*).$$

To show that upon the satisfying (45) and the cardinality constraint, product k must be in A_n^* if it satisfies (46). Suppose that A_n^* does not contain k. Then by the optimality of A_n^*,

$$\frac{R^* + r_{n,k}v_{n,k}^w}{H(V^* + v_{n,k}^w)} - \frac{R^*}{H(V^*)} = \frac{v_{n,k}^w}{H(V^* + v_{n,k}^w)} \left(r_{n,k} - \frac{R^*}{H(V^*)} \frac{H(V^* + v_{n,k}^w) - H(V^*))}{v_{n,k}^w} \right) < 0,$$

which contradicts (46) since by the concavity of $H(x)$,

$$\frac{H(V^* + v_{n,k}^w) - H(V^*)}{v_{n,k}^w} < H'(V^*).$$

$\square$

Based on possible values of $g_n^* H'(E_n^* + E_{-n})(v_{n,k}^w - v_{n,j}^w)$, there can be $\mathcal{O}(|S_n|^2)$ of candidate assortments that satisfy the conditions in the theorem, and the optimal assortment can be determined by comparing profits among these candidates. The procedure in Table 6 specifies the process for deriving the optimal solution.

In this procedure, steps 1-4 generate an initial assortment by assuming that

$$g_n^* H'(E_n^* + E_{-n}) < s_{jk} \text{ for all } s_{jk} > 0 \ (j, k \in S_n) \quad \text{and} \quad g_n^* H'(E_n^* + E_{-n}) < r_{n,k} \text{ for all } k \in S_n.$$

Step 5 sets up a procedure for increasing the assumed value of $g_n^* H'(E_n^* + E_{-n})$, and Step 6 updates the assortment accordingly. Step 6.1 corresponds to the point where the latter value increases to the point where (24) replaces (23) so product j should be ranked higher than product k, and Step 6.2 corresponds to the point where product k is ranked lower than the dummy product and thus should not be in the assortment even the cardinality constraint allows it. The resulting assortment determined at Step 6.3 becomes the new assortment if it generates a higher profit than the default one.

1. For each product pair $j, k \in S_n$ with $v^w_{n,k} > v^w_{n,j}$, use (25) to calculate s_{jk}.

2. For each product pair $j, k \in S_n$ with $v^w_{n,k} > v^w_{n,j}$, rank k higher than j if $s_{jk} > 0$ and j higher than k if $s_{jk} = 0$. For each product pair $j, k \in S_n$ with $v^w_{n,k} = v^w_{n,j}$, rank k higher than j if $r_{n,k} > r_{n,j}$ and j higher than k if $r_{n,k} < r_{n,j}$. Break the tie arbitrary if $r_{n,k} = r_{n,j}$.

3. Construct an completely-ordered list of all products, denoted by $\mathcal{L}$, where a product has a higher order than all those that ranked lower in step 2.

4. Select $C_n \wedge |\mathcal{L}|$ highest ordered products from $\mathcal{L}$ as the default optimal assortment A^*_n and use (22) to calculate $g^*_n = g_n(A^*_n)$.

5. Arrange all s_{jk} $(j, k \in S_n, s_{jk} > 0)$ and $r_{n,j}$ in a list of ascending order. Let m be the number of entries s_{jk} on that list, denote the entries by σ_i, $i \in \{1, 2, \cdots, m + |S_n|\}$ (so σ_i is either s_{jk} for some $j, k \in S_n$ or $r_{n,k}$ for some $k \in S_n$).

6. For $i \in \{1, 2, \cdots m + |S_n|\}$:

   6.1. if $\sigma_i = s_{jk}$ for some j, k, exchange positions j and k on list $\mathcal{L}$.

   6.2. if $\sigma_i = r_{n,k}$ for some k, remove product k from $\mathcal{L}$.

   6.3. select highest-ordered $C_n \wedge |\mathcal{L}|$ products from $\mathcal{L}$ to form a candidate assortment A_n. Use (22) to calculate $g_n(A_n)$.

   6.4. if $g_n(A_n) > g^*_n$, then let $A_n \to A^*_n$ and $g^*_n = g_n(A_n)$.

Table 6: Unified Solution Procedure