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Assemble-to-Order Inventory Management via Stochastic Programming: Chained BOMs and the M-System

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We study an inventory management mechanism that uses two stochastic programs (SPs), the customary one-period assemble-to-order (ATO) model and its relaxation, to conceive control policies for dynamic ATO systems. We introduce a class of ATO systems, those that possess what we call a "chained BOM." We prove that having a chained BOM is a sufficient condition for both SPs to be $L^{\natural}$ convex in the first-stage decision variables. We show by examples the necessity of the condition. For ATO systems with a chained BOM, our result implies that the optimal integer solutions of the SPs can be found efficiently, and thus expedites the calculation of control parameters. The M system is a representative chained BOM system with two components and three products. We show that in this special case, the SPs can be solved as a one-stage optimization problem. The allocation policy can also be reduced to simple, intuitive instructions, of which there are four distinct sets, one for each of four different parameter regions. We highlight the need for component reservation in one of these four regions. Our numerical studies demonstrate that achieving asymptotic optimality represents a significant advantage of the SP-based approach over alternative approaches. Our numerical comparisons also show that outside of the asymptotic regime, the SP-based approach has a commanding lead over the alternative policies. Our findings indicate that the SP-based approach is a promising inventory management strategy that warrants further development for more general systems and practical implementations.

Key words: assemble-to-order; stochastic programming; discrete convexity; bill of materials; asymptotic optimality
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1. Introduction

Many inventory control problems remain unsolved despite years of effort. The assemble-to-order (ATO) inventory control problem is one of these. ATO is an effective manufacturing strategy to deal with demand uncertainty and thus has been widely adopted in practice. This prevalence in practice has triggered much academic interest. Although some special cases have been solved, notably the single product problem (solved by Rosling 1989) and the single period problem (solved by Song and Zipkin 2003), the general multi-period multi-product problem has not yet been solved. A typical approach taken towards this problem in the literature, epitomized by Lu and Song (2005), is to find a set of policies within which a performance analysis can be carried out and to optimize within this set of policies (see also Agrawal and Cohen 2001, Akcay and Xu 2004, Hausman et al. 1998, Zhang 1997). If the chosen set of policies contains the overall optimal policy then the approach will find the optimal policy. However, if the chosen set does not contain the optimal policy then the approach cannot find the optimal policy, and if none of the policies in the chosen set are very good, the optimal one among them cannot be very good.

Doğru et al. (2010) took a different approach to this problem, which has been further developed in Reiman and Wang (2012) and Reiman and Wang (2015). The approach boils down to the following three steps: (i) by 'relaxing' certain constraints present in the original inventory system, obtain a stochastic program (SP)

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whose solution provides a lower bound on the cost achievable by any inventory control policy; (ii) solve the SP; and (iii) 'translate' the SP solution into an implementable replenishment policy that determines the amounts of various components to order and an allocation policy that determines the amounts of different products to serve at each point of time.

For systems with identical lead times, this approach leads to a base stock replenishment policy where the base stock levels are set by solving a two-stage SP. The SP is also used to define a general allocation procedure (Reiman and Wang 2015), which makes decisions by re-solving the second-stage problem with state-dependent inputs over time. Reiman and Wang (2015) justifies this approach by proving that it is asymptotically optimal, that is, as the lead time grows large, the percentage difference between the resulting long-run average inventory cost and its minimum converges to 0. Since long lead times imply high demand uncertainty, and high inventory costs, satisfying this asymptotic optimality criterion addresses the most significant piece of the problem.

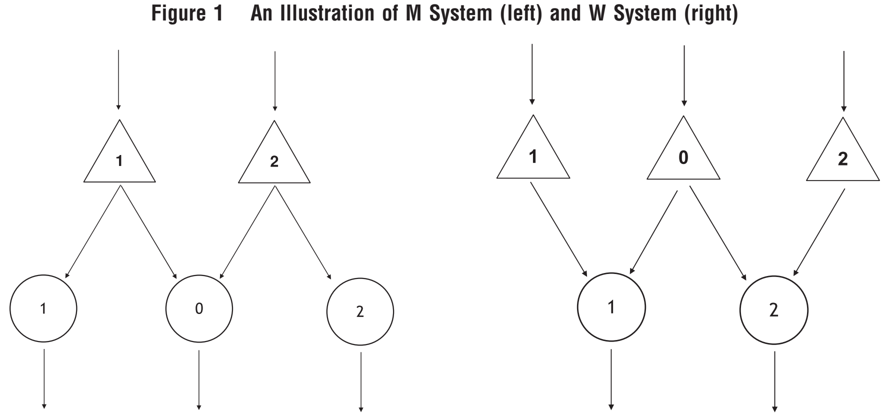
This promising new approach has several important issues that remains to be resolved. This paper addresses two issues: structural properties that simplify the problem and numerical testing for performance evaluation. Part of our analysis of the first subject applies to general ATO systems with identical lead times. The rest focuses exclusively on the M system shown in Figure 1. The system has three products ($i = 0, 1, 2$) and two components ($j = 1, 2$). A unit of component i can be used either separately to build product i ($i = 1, 2$) or jointly with a unit of the other component to assemble product 0. The M system is a template model for manufacturing bundled products and provisioning maintenance parts, and thus is a prominent special case of ATO inventory models and has been widely used as a test bed for evaluating ATO inventory policies (e.g., Lu and Song 2005, Lu et al. 2010, Nadar et al. 2014).

1.1. Structure and Simplification

The SP-based approach involves solving two SPs: the one-period ATO model introduced in Song and Zipkin (2003) and the relaxation of that SP where all non-negativity constraints are removed. Both SPs have two stages with linear objective functions and constraints. The first-stage problem optimizes component ordering and the second-stage problem optimizes component allocation. Both decisions are made in integral units, so the first-stage problem is a discrete optimization problem and the second-stage problem is an integer linear program (ILP). The solution procedure can be greatly simplified if the objective function of the SP is $L^\natural$ convex in the first-stage decision variables (Murota 1998). In this case, one can use the steepest decent search to find a local optimum, which is guaranteed to be a globally optimal solution (Murota and Shioura 2014).

There have been a few studies on $L^\natural$ convexity of an inventory, revenue management, or pricing model (see e.g., Chen et al. 2014, 2016, Huh and Janakiraman 2010, Lu and Song 2005, Pang et al. 2012, Simchi-Levi et al. 2014, Zipkin 2008, 2016). Most relevant to this study is the discussion in Zipkin (2016) of the same one-period ATO model we consider here. It focuses on a particular ATO system structure, the tree family. An example of such system is the W system in Figure 1, where two products 1 and 2 share a common component 0 while each also uses a separate component, 1 and 2, respectively. The analysis shows that for any system in the tree family, under the optimal choice of the first-stage decision variables, the constraint set of the second stage ILP forms a polymatroid. As a result, the SP objective function is "cover-$L^\natural$ convex", a property that, while weaker than $L^\natural$ convex, can still be exploited to simplify the solution procedure. Zipkin (2016) observes that the M system does not fit into the polymatroid characterization, and yet both the one-period ATO model and its relaxation are $L^\natural$ convex in this case. So he asks "Is some other principle at work here? If so, perhaps it can be applied to other family structures."

Figure 1 An Illustration of M System (left) and W System (right)



This interesting question inspires a major contribution of this paper, which is to identify a new family of ATO structures, the "chained-BOM." The M system is the simplest multi-product system in that family. We prove that for systems with this structure, both SPs, the one-period ATO model and its relaxation, are $L^\natural$ convex everywhere in the domain of first-stage decision variables and under any demand distribution. We also show by examples that for systems that do not fit into that category, the objective function of the one-period ATO system is generally not $L^\natural$ convex.

We also demonstrate that in case of the M system, the search for the optimal SP solutions can be further simplified. Not only can the number of search steps be cut down drastically because of $L^\natural$ convexity, but also the complexity of evaluating the objective value at each step can be greatly reduced. By deriving the explicit solution of the second-stage ILP, we transform the two-stage SPs into one-stage optimization problems of which the objective function has desirable properties. Moreover, we show that this development also facilitates the implementation of the dynamic inventory control policies. Rather than allocating components by re-solving the second-stage problems repeatedly over time, which is the case with general ATO systems, in the M system, the same procedure can be carried out by following simple rules. Depending on cost parameters, the rule can be that of static priority, state-dependent priority, or reservation. All of them are intuitive and none of them require computational effort.

1.2. Performance Evaluation

Regarding performance evaluation, when exact optimality is beyond reach, attaining the aforementioned asymptotic optimality, as is proved by Reiman and Wang (2015), is highly desirable. In a study of related ATO inventory/production systems, Plambeck and Ward (2006) also apply the same criterion to develop control policies. However, one may ask whether this is an easy target that can be attained by many other methods. If so, then asymptotic optimality is not an effective filter that can fully support the use and continuing investigation of the SP-based approach (e.g., generalizing the approach to systems with general lead times, as in Reiman and Wang 2012). Moreover, asymptotic optimality does not guarantee optimality in all cases, so one may also want to know how the SP-based approach compares with alternative policies in non-asymptotic parameter regimes. This paper carries out an extensive numerical study to address both questions.

We provide evidence leading us to conjecture that prevailing approaches in the literature, namely, a base stock replenishment policy combined with FIFO allocation (see e.g., Lu and Song 2005) or a No Holdback (NHB) allocation scheme (see e.g., Lu et al. 2010, Song and Zhao 2009), are not asymptotically optimal. In addition, we also consider a priority allocation policy, which is a better implementation of the NHB principle than the first-ready-first-serve (FRFS) policy in Lu et al. (2010). Our numerical evidence, corroborated by discussions in previous papers, leads us to conjecture that even this improved approach fails to be asymptotically optimal in some parameter regions.

We also show that the SP-based approach dominates other policies outside the asymptotic regime, that is, systems with short lead times. By isolating policy differences to make controlled comparisons, we illustrate that the advantage of the SP-based approach can be attributed to both its way of setting base stock levels and its rules for allocating components.

The rest of the paper is organized as follows. In section 2, we formulate the ATO model and briefly review the development of the SP-based approach to put our work into perspective. In section 3 we introduce the notion of a chained BOM and prove that having a chained BOM is a sufficient condition for the two SPs to have optimal values that are $L^\natural$ convex. We analyze the M system in section 4, followed by numerical studies in section 5. Concluding remarks are given in section 6. The Appendix contains proofs and other technical details.

As for relevant notation, $\mathbb{Z}^n$ is the space of n-dimensional integer vectors, $\mathbb{Z}^n_+$ is the space of n-dimensional non-negative integer vectors (omitting the superscript if $n = 1$), and $\mathbf{1}$ is the vector of all 1s. We will also use operators $\wedge$ and $\vee$ to denote taking the minimum and the maximum of two numbers respectively.

2. Problem Description and Previous Results

We will first introduce the formulation of our ATO model in section 2.1 and then briefly review the SP-based approach in section 2.2.

2.1. ATO Model Formulation

We consider a multi-period ATO system that has m products and n components. A unit of product i $(1 \le i \le m)$ is assembled from a_{ji} units of component j $(1 \le j \le n)$. The matrix $A = \{a_{ji}, 1 \le j \le n, 1 \le i \le m\}$ is thus the bill of materials (BOM). The system is controlled by a replenishment policy γ, which determines when and how many parts to order, and an allocation policy p, which determines how to distribute available parts to different products. For the sake of brevity, we focus our discussion on the continuous-review model while noting that the same analysis applies to the periodic-review model.

We assume that all parts have a common replenishment lead time L. Demands for all products arrive according to a compound Poisson process $\mathcal{D} = \{\mathcal{D}(t),\, t \geq 0\}$, where $\mathcal{D}(t) = (\mathcal{D}_1(t), \ldots, \mathcal{D}_m(t))$, $t \geq 0$, and $\mathcal{D}_i(t)$ is the amount of demand for product i $(1 \leq i \leq n)$ that arrives within the interval $[0, t]$. Denote the demand that arrives during the period $(t_1, t_2]$, $0 \leq t_1 < t_2$, by

$$\mathbf{D}(t_1, t_2) = \mathcal{D}(t_2) - \mathcal{D}(t_1).$$

With a slight abuse of notation, we let $\mathbf{D}(t) = \mathbf{D}(t - L, t)$ denote the demand that arrives within the lead time ending at $t \geq L$. Note that $\mathbf{D}(t)$ has the same distribution for all $t \geq L$, and let $\mathbf{D} = (D_1, \ldots, D_m)$ denote a random vector that has this distribution.

At each moment, the inventory manager observes new demand arrivals and receives previously-ordered components, if any. Afterwards, she decides how to allocate available components to serve demands and whether to order components, and if so how many. Unserved demand is backlogged and unused components stay in inventory. Define $Z_i(t_1, t_2)$, $0 \leq t_1 < t_2$, to be the amount of product i demand $(1 \leq i \leq m)$ served during the period $(t_1, t_2]$, and let $\mathbf{Z}(t_1, t_2) = (Z_1(t_1, t_2), \ldots, Z_m(t_1, t_2))$. With $B_i(t)$ denoting the backlog of product i at time $t \geq 0$ and $\mathbf{B}(t) = (B_1(t), \ldots, B_m(t))$ for $0 \leq t_1 < t_2$, the backlog level satisfies

$$\mathbf{B}(t_2) = \mathbf{B}(t_1) + \mathbf{D}(t_1, t_2) - \mathbf{Z}(t_1, t_2). \tag{1}$$

Let $R_j(t_1, t_2)$ denote the amount of component j $(1 \leq j \leq n)$ replenishment ordered during the period $(t_1, t_2]$, $-L \leq t_1 < t_2$, and $\mathbf{R}(t_1, t_2) = (R_1(t_1, t_2), \ldots, R_n(t_1, t_2))$. With $I_j(t)$ denoting the inventory level of component j at time $t \geq 0$ and $\mathbf{I}(t) = (I_1(t), \ldots, I_n(t))$ for $0 \leq t_1 < t_2$, the inventory level satisfies

$$\mathbf{I}(t_2) = \mathbf{I}(t_1) + \mathbf{R}(t_1 - L, t_2 - L) - A\mathbf{Z}(t_1, t_2). \tag{2}$$

Let b_i $(1 \leq i \leq m)$ be product i's backlog cost per unit of time, let h_j $(1 \leq j \leq n)$ be component j's inventory holding cost per unit of time, let $\mathbf{b} = (b_1, \ldots, b_m)$, $\mathbf{h} = (h_1, \ldots, h_n)$, and let $\mathbf{c} = \mathbf{b} + A^T\mathbf{h}$. We define c_i as the unit inventory cost, the value of which determines how much inventory cost can be removed from the system by serving one unit of product i $(i = 1, \ldots, m)$. For instance, in the W system, $c_i = b_i + h_0 + h_i$ $(i = 1, 2)$. One may assume without loss of generality that $c_1 \geq c_2$, in which case serving product 1 is more desirable than serving product 2. In the M system, $c_0 = b_0 + h_1 + h_2$, $c_1 = b_1 + h_1$, $c_2 = b_2 + h_2$. Although we can (and do) assume, without loss of generality, that $c_1 \geq c_2$, nonetheless there are four cost parameter regions:

Region A	Region B	Region C	Region D
$c_1+c_2<c_0$	$c_2 \leq c_1 < c_0 \leq c_1+c_2$	$c_2 < c_0 \leq c_1$	$c_0 \leq c_2 \leq c_1$

As will be evident in our discussion below, different allocation policies apply for different regions.

The goal of ATO inventory management is to find a feasible replenishment policy γ and an allocation policy p to minimize the long-run average expected total inventory cost

$$\mathcal{C}^{\gamma,p} \equiv \limsup_{T\to\infty} \frac{1}{T}\int_0^T E[\mathbf{b}\cdot\mathbf{B}(t) + \mathbf{h}\cdot\mathbf{I}(t)]dt. \tag{3}$$

To be feasible, a policy cannot serve more demand than the amount arrived or the amount allowed by the available supply of required components. In addition, the policy needs to be non-anticipating, that is, it can depend only on information available at the moment any decision is made. See Doğru et al. (2010) or Reiman and Wang (2015) for more details.

2.2. The SP-based Approach

Doğru et al. (2010) developed a SP-based approach to address ATO inventory systems with identical lead times. Their policy development focuses on the W system. Reiman and Wang (2015) generalized this approach to systems with general BOMs and proved it satisfies an asymptotic optimality criterion. Their proposal and conclusions are summarized as follows:

1. *Lower Bound:* The cost objective (3) is bounded from below by the optimal solution of the following two-stage SP

$$\hat{C}^* = \inf_{\mathbf{y}} \hat{C}(\mathbf{y}) \tag{4}$$

$$\begin{aligned} &\text{where } \hat{C}(\mathbf{y}) \equiv \mathbf{b}\cdot E[\mathbf{D}] + \mathbf{h}\cdot\mathbf{y} - E[\varphi(\mathbf{y};\mathbf{D})] \\ &\text{and } \varphi(\mathbf{y};\mathbf{d}) = \max_{\mathbf{z}}\{\mathbf{c}\cdot\mathbf{z} | \mathbf{z} \leq \mathbf{d}, A\mathbf{z} \leq \mathbf{y}\},\ \mathbf{d} \in \mathbb{Z}_+^m. \end{aligned} \tag{5}$$

Here parameters $\mathbf{b}$, $\mathbf{h}$, and $\mathbf{c}$ are defined the same as those in section 2.1, and $\mathbf{D}$ is a random variable that has the same distribution as the lead time demand $\mathbf{D}(t)$ $(t \geq 0)$. The SP is a relaxed version (with the removal of non-negativity constraint) of the following one-period ATO model considered in Song and Zipkin (2003):

$$C^* = \min_{\mathbf{y}\geq 0} C(\mathbf{y}) \tag{6}$$

$$\begin{aligned} &\text{where } C(\mathbf{y}) \equiv \mathbf{b}\cdot E[\mathbf{D}] + \mathbf{h}\cdot\mathbf{y} - E[\phi(\mathbf{y};\mathbf{D})] \\ &\text{and } \phi(\mathbf{y};\mathbf{d}) = \max_{\mathbf{z}\geq 0}\{\mathbf{c}\cdot\mathbf{z} | \mathbf{z} \leq \mathbf{d}, A\mathbf{z} \leq \mathbf{y}\}, \mathbf{d} \in \mathbb{Z}_+^m, \end{aligned} \tag{7}$$

which is also a simple version of the 'newsvendor network' of Harrison and Van Mieghem (1999).

2. *Component Replenishment:* Components are ordered according to a base stock policy where the

base stock level is the weighted average of an optimal solution of the SP (4), $\hat{\mathbf{y}}^*$, and that of SP (6), $\mathbf{y}^*$. The weight can be any value between 0 and 1 without compromising asymptotic optimality that we will discuss below.

3. *Component Allocation:* At each time t ($t \geq 0$), set backlog targets

$$\mathbf{B}^*(t) = \arg\min\{\mathbf{c}\cdot\mathbf{B}|\mathbf{B}\geq\mathbf{0}, A\mathbf{B}\geq\mathbf{Q}(t)\}\text{for all } t\geq L. \tag{8}$$

where $\mathbf{Q}(t)$ represents component shortage at time t, given by

$$\mathbf{Q}(t) \equiv A\mathbf{B}(t) - \mathbf{I}(t) = A\mathbf{D}(t) - \mathbf{y}^*, \quad t\geq L. \tag{9}$$

Observe that the second equality is an implication of a base stock policy with $\mathbf{y}^*$ as the base stock levels. Since $\mathbf{D}(t) \stackrel{d}{=} \mathbf{D}$, (8) is analogous to the recourse LP (6), where we transform the latter by defining $\mathbf{Q} = A\mathbf{D} - \mathbf{y}^*$ as component shortage and use $\mathbf{B} = A\mathbf{D} - \mathbf{z}$ as control variables. The allocation can be carried out by any policy that satisfies the following Allocation Principle:

- Do not serve any demand i at time t if its current backlog level $B_i(t)$ does not exceed the target $B_i^*(t)$ $1 \leq i \leq m$.
- For all other demands with $B_i(t) > B_i^*(t)$ ($1 \leq i \leq m$) use all available components to clear as much excess ($B_i(t) - B_i^*(t)$) as possible, following any sequence of serving demand.

4. The application of this approach is justified by its asymptotic optimality. Specifically, let there be a family of ATO systems with a general BOM and indexed by the common lead time of all components, L. All parameters other than L are held fixed, while $L \to \infty$. Let $\mathcal{C}_L^{(\gamma,p)}$ denote the long run average cost for policy (γ, p), and let $\underline{\mathcal{C}}_L^*$ denote the lower bound, both for lead time L. Let $\{(\gamma^*(L), p^*(L)), L > 0\}$ denote a family of policies that use base-stock replenishment with base-stock levels $\mathbf{y}^{(L)*}$ and use an allocation policy that satisfies the allocation principle. Then

$$\lim_{L\to\infty} \frac{\mathcal{C}_L^{\gamma^*(L),p^*(L)}}{\underline{\mathcal{C}}_L^*} = 1. \tag{10}$$

3. $L^\natural$ Convexity of Stochastic Programs

As defined in Murota (2003a), an integer-valued function $f(\mathbf{y})$ is $L^\natural$ convex if

$$f(\mathbf{y}^a)+f(\mathbf{y}^b)\geq f([\mathbf{y}^a-\alpha\mathbf{1}]\vee\mathbf{y}^b)+f(\mathbf{y}^a\wedge[\mathbf{y}^b+\alpha\mathbf{1}]) \tag{11}$$

for all $\mathbf{y}^a$, $\mathbf{y}^b \in \mathbb{Z}^n$ and $\alpha \in \mathbb{Z}_+$. If $C(\mathbf{y})$ is $L^\natural$ convex in $\mathbf{y}$, then the SP (6) can be solved by using a steepest-decent algorithm to find a local minimum (Murota 2003b). The same applies to $\hat{C}(\mathbf{y})$.

To understand when $L^\natural$ convexity applies, we first introduce a property satisfied by some ATO systems that we call *chained BOM*, in section 3.1. We then prove in section 3.2 that for any system that has this property, both $\hat{C}(\mathbf{y})$ and $C(\mathbf{y})$ are $L^\natural$ convex. In section 3.3, we show that the property may not hold for other system.

3.1. Definition of Chained BOM

We define that an ATO system has a chained-BOM if: (i) all elements in BOM matrix A take binary values, in which case each product uses either one or zero unit of a component, and (ii) for any two columns i and i' of A

$$\begin{aligned}&\text{if } \sum_{j=1}^n a_{ji}a_{ji'} \geq 1, \text{ then}\\ &\text{either } a_{ji}\geq a_{ji'} \text{ for all } j=1,\ldots,n, \text{ or } a_{ji}\leq a_{ji'}\\ &\quad\text{for all } j=1,\ldots,n,\end{aligned} \tag{12}$$

which means that if two products share a common component, then the set of components used by one of these products must contain that of the other.

For an ATO system with a chained BOM, let $\mathcal{S}$ be the collection of all component sets for assembling a product. For each $s \in \mathcal{S}$, define $p(s) \in \mathcal{S}$ as its proper superset with the smallest number of elements. Since any superset of s shares components with $p(s)$, by (12), it must be either $p(s)$ or a larger set that contains $p(s)$. Hence for each $s \in \mathcal{S}$, $p(s)$ is unique if it exists.

Denote $\bar{s} = \{1, \ldots, n\}$ as the set of all components. Note that $p(\bar{s})$ does not exist. On the other hand, $p(s)$ does exist for every $s \in \mathcal{S} \setminus \{\bar{s}\}$ if $\bar{s} \in \mathcal{S}$. Assuming the latter is without the loss of generality: if $\bar{s} \notin \mathcal{S}$, then (12) implies that either there are redundant components that are not used by any product ($a_{ji} = 0$ for all i) or A contains a subset of columns $\mathcal{I}$ such that

$$\sum_{j=1}^n a_{ji}a_{ji'} = 0 \text{ for any } i\in\mathcal{I} \text{ and } i'\notin\mathcal{I}.$$

Thus the ATO system can be divided into two non-overlapping subsystems that can be analyzed separately.

For each $s \in \mathcal{S}$, define the following chain of inclusion:

$$\mathbb{C}(s) = \{s, p(s), p(p(s)), \ldots, \bar{s}\}. \tag{13}$$

Denote by $\kappa(s)$ the set of products that use exactly those components in s. Then $\{\kappa(s), s \in \mathcal{S}\}$ is a partitioning of the product set, that is, $\kappa(s)$ are mutually exclusive and

$$\bigcup_{s\in\mathcal{S}}\kappa(s)=\{1,\ldots,m\}.$$

Denote by $\mathcal{K}(s)$ the set of products that use all components in s ($s\in\mathcal{S}$). Then

$$\mathcal{K}(s)=\bigcup_{s'\in\mathbb{C}(s)}\kappa(s').$$

As an example,

$$A=\begin{pmatrix}1&0&1&1&1&1\\0&0&1&1&1&1\\0&0&1&1&1&1\\0&1&0&0&1&1\\0&1&0&0&1&1\end{pmatrix}$$

is a chained BOM. Here $\mathcal{S}=\{s_1,s_2,s_3,s_4\}$, where

$$\begin{aligned}&s_1=\{1\},s_2=\{4,5\},s_3=\{1,2,3\},\\&s_4(=\bar{s})=\{1,2,3,4,5\},\\&p(s_1)=s_3,p(s_2)=p(s_3)=s_4,\\&\kappa(s_1)=\{1\},\kappa(s_2)=\{2\},\kappa(s_3)=\{3,4\},\\&\kappa(s_4)=\{5,6\},\\&\mathcal{K}(s_1)=\{1,3,4,5,6\},\mathcal{K}(s_2)=\{2,5,6\},\\&\mathcal{K}(s_3)=\{3,4,5,6\},\mathcal{K}(s_4)=\{5,6\},\\&\mathbb{C}(s_1)=\{s_1,s_3,s_4\},\mathbb{C}(s_2)=\{s_2,s_4\},\\&\text{and }\mathbb{C}(s_3)=\{s_3,s_4\},\mathbb{C}(s_4)=\{s_4\}.\end{aligned}$$

Remark 1. The nested ATO systems defined in ElHafsi et al. (2008) is a special case of the chained BOM structure. In the former case, $\mathcal{S}$ forms a single chain of inclusion, excluding systems that have two products assembled from two separate sets of components. It also requires that $|\kappa(s)|=1$, so no two products with distinct backlog costs can be assembled from the same set of components. The chained BOM generalizes the nested structure by removing both restrictions.

Remark 2. The chained BOM is related to but not the same as the tree family defined in Zipkin (2016). The M system in Figure 1 is an example of the former and the W system is an example of the latter. Their BOM matrices A are transposes of each other. With a chained BOM, no two products can share a common component while each also uses a separate component of its own. With a tree family, no two components can be used as a bundle in one product while each is also used separately in different products.

3.2. $L^\natural$ Convexity: Theorem and Proof

From the definition (11), one can easily verify that all linear functions are $L^\natural$ convex. Moreover, if $f(\mathbf{y};\mathbf{d})$ satisfies (11) for all $\mathbf{d}\in\mathbb{Z}^m_+$, then $\mathbf{E}[f(\mathbf{y};\mathbf{D})]$ is $L^\natural$ convex. Therefore in the one-period ATO model (6), if the optimal value of the second stage ILP, which represents the maximum reduction of the inventory cost by the allocation decision, is a $L^\natural$ concave function of the order quantities determined at the first stage, i.e., for any $\mathbf{y}^a,\mathbf{y}^b\in\mathbb{Z}^n_+$, $\alpha\in\mathbb{Z}_+$, and $\mathbf{d}\in\mathbb{Z}^m_+$,

$$\begin{aligned}\phi(\mathbf{y}^a;\mathbf{d})+\phi(\mathbf{y}^b;\mathbf{d})\le{}&\phi((\mathbf{y}^a-\alpha\mathbf{1})\vee\mathbf{y}^b;\mathbf{d})\\&+\phi(\mathbf{y}^a\wedge(\mathbf{y}^b+\alpha\mathbf{1});\mathbf{d}),\end{aligned}\tag{14}$$

then $C(\mathbf{y})$ is $L^\natural$ convex. The theorem below shows that having a chained BOM is a sufficient condition for the above.

Theorem 1. *For any system with a chained BOM,* (14) *holds for any values of* $\mathbf{y}^a,\mathbf{y}^b\in\mathbb{Z}^n_+$, $\alpha\in\mathbb{Z}_+$, *and* $\mathbf{d}\in\mathbb{Z}_m^+$. *Thus* $C(\boldsymbol{y})$ *is* $L^\natural$ *convex everywhere and under any demand distribution, i.e.,*

$$C(\mathbf{y}^a)+C(\mathbf{y}^b)\ge C((\mathbf{y}^a-\alpha\mathbf{1})\vee\mathbf{y}^b)+C(\mathbf{y}^a\wedge(\mathbf{y}^b+\alpha\mathbf{1}))\tag{15}$$

To prove the theorem, we will make use of the following simple facts:

1. For any two integers x_1 and x_2,
$$x_1+x_2=x_1\wedge x_2+x_1\vee x_2.\tag{16}$$
2. For any integers x_1, x_2, x_3, and x_4,
$$\begin{aligned}&(x_1-x_4)\wedge(x_2-x_3)\le x_1\wedge x_2-x_3\wedge x_4\\&\quad\le(x_1-x_4)\vee(x_2-x_3),\\&\text{and }(x_1-x_4)\wedge(x_2-x_3)\le x_1\vee x_2-x_3\vee x_4\\&\quad\le(x_1-x_4)\vee(x_2-x_3).\end{aligned}\tag{17}$$
3. Consider the following family of ILPs parameterized by an integer x, with $\mathbf{w}\in\mathbb{Z}^k_+$ and $k\in\mathbb{Z}_+$ fixed, and $c_i\ge 0$ $(1\le i\le k)$,
$$\zeta(x)=\max_{\mathbf{z}\in\mathbb{Z}^k_+}\left\{\sum_{i=1}^k c_iz_i\,\middle|\,\sum_{i=1}^k z_i\le x,z_i\le w_i,1\le i\le k\right\}.\tag{18}$$

The ILP optimally allocates x units of a resource to a set of candidates, where w_i, c_i are respectively demand and unit value of candidate i $(1\le i\le k)$. Without loss of generality, assume $c_1\ge c_2\cdots\ge c_k$. Then the optimal solution is the greedy one:

$$z_i^*=\min(w_i,(x-\sum_{i'=1}^{i-1}w_{i'})^+),i=1,\ldots,k,$$

where an empty sum is 0. For any integer $x\ge 0$,

$$\Delta\zeta(x) \equiv \zeta(x+1) - \zeta(x) = \begin{cases} c_1 & \text{if} \quad x < w_1 \\ c_l & \text{if} \quad \sum_{i=1}^{l-1} w_i \le x < \sum_{i=1}^{l} w_i \ (1 \le l \le k) \\ 0 & \text{if} \quad \sum_{i=1}^{k} w_i \le x \end{cases}.$$

Hence $\Delta\zeta(x)$ decreases in x, which means that $\zeta(x)$ is a discretely-concave function. As an implication, for any integers x_i $(i = 1, 2, 3, 4)$ where $x_1 \le x_2 \le x_4$, $x_1 \le x_3 \le x_4$, and $x_1 + x_4 = x_2 + x_3$,

$$\zeta(x_1) + \zeta(x_4) \le \zeta(x_2) + \zeta(x_3). \tag{19}$$

Proof. To prove (14) under a fixed demand vector $\mathbf{d} \in \mathbb{Z}_+^m$, let A be a binary matrix that satisfies (12). For any given $\mathbf{y}^a, \mathbf{y}^b \in \mathbb{Z}_+^n$, denote $\mathbf{z}^a = (z_1^a, \ldots, z_m^a)$ and $\mathbf{z}^b = (z_1^b, \ldots, z_m^b)$ as an optimal solution of $\phi(\mathbf{y}^a, \mathbf{d})$ and $\phi(\mathbf{y}^b, \mathbf{d})$ respectively.

In solutions $\mathbf{z}^a$ and $\mathbf{z}^b$ respectively, let

$$Z_s^a = \sum_{i\in\kappa(s)} z_i^a \text{ and } Z_s^b = \sum_{i\in\kappa(s)} z_i^b \tag{20}$$

be total demands served for products in $\kappa(s)$, the set of products that use exactly all components in s $(s \in \mathcal{S})$. Similarly, in these two solutions, let

$$\bar{Z}_s^a = \sum_{i\in\mathcal{K}(s)} z_i^a \text{ and } \bar{Z}_s^b = \sum_{i\in\mathcal{K}(s)} z_i^b \tag{21}$$

be total demands served for products in $\mathcal{K}(s)$, the set of products that use all components in s $(s \in \mathcal{S})$. By the definition and existence of $p(s)$ (for $s \in \mathcal{S} \setminus \{\bar{s}\}$),

$$\kappa(s) = \mathcal{K}(s) \setminus \mathcal{K}(p(s)) \text{ for all } s \in \mathcal{S} \setminus \{\bar{s}\} \text{ and } \kappa(\bar{s}) = \mathcal{K}(\bar{s}).$$

For convenience, define $\bar{Z}_{p(\bar{s})}^a = \bar{Z}_{p(\bar{s})}^b = 0$. Then

$$Z_s^a = \bar{Z}_s^a - \bar{Z}_{p(s)}^a \text{ and } Z_s^b = \bar{Z}_s^b - \bar{Z}_{p(s)}^b, \ s \in \mathcal{S}. \tag{22}$$

For any $\alpha \in \mathbb{Z}_+$ and $s \in \mathcal{S}$, let

$$\begin{aligned} Z_s' &= (\bar{Z}_s^a - \alpha) \vee \bar{Z}_s^b - (\bar{Z}_{p(s)}^a - \alpha) \vee \bar{Z}_{p(s)}^b \\ \text{and } Z_s'' &= (\bar{Z}_s^a - \alpha) \wedge \bar{Z}_s^b - (\bar{Z}_{p(s)}^a - \alpha) \wedge \bar{Z}_{p(s)}^b \\ &= \bar{Z}_s^a \wedge (\bar{Z}_s^b + \alpha) - \bar{Z}_{p(s)}^a \wedge (\bar{Z}_{p(s)}^b + \alpha). \end{aligned}$$

Then following (16) and (22),

$$\begin{aligned} Z_s' + Z_s'' =& [(\bar{Z}_s^a - \alpha) \vee \bar{Z}_s^b + (\bar{Z}_s^a - \alpha) \wedge \bar{Z}_s^b] \\ & - [(\bar{Z}_{p(s)}^a - \alpha) \vee \bar{Z}_{p(s)}^b + (\bar{Z}_{p(s)}^a - \alpha) \wedge \bar{Z}_{p(s)}^b] \\ =& [(\bar{Z}_s^a - \alpha) + \bar{Z}_s^b] - [(\bar{Z}_{p(s)}^a - \alpha) + \bar{Z}_{p(s)}^b] \\ =& \bar{Z}_s^a + \bar{Z}_s^b - \bar{Z}_{p(s)}^a - \bar{Z}_{p(s)}^b, \\ =& Z_s^a + Z_s^b, s \in \mathcal{S}, \end{aligned} \tag{23}$$

and following (17),

$$\begin{aligned} &(\bar{Z}_s^a - \bar{Z}_{p(s)}^a) \wedge (\bar{Z}_s^b - \bar{Z}_{p(s)}^b) \le Z_s' \le (\bar{Z}_s^a - \bar{Z}_{p(s)}^a) \vee (\bar{Z}_s^b - \bar{Z}_{p(s)}^b) \\ \text{and } &(\bar{Z}_s^a - \bar{Z}_{p(s)}^a) \wedge (\bar{Z}_s^b - \bar{Z}_{p(s)}^b) \le Z_s'' \le (\bar{Z}_s^a - \bar{Z}_{p(s)}^a) \vee (\bar{Z}_s^b - \bar{Z}_{p(s)}^b), \quad s \in \mathcal{S}. \end{aligned}$$

Using (22) to replace $\bar{Z}$s in the above,

$$Z_s^a \wedge Z_s^b \le Z_s' \le Z_s^a \vee Z_s^b \text{ and } Z_s^a \wedge Z_s^b \le Z_s'' \le Z_s^a \vee Z_s^b, \ s \in \mathcal{S}. \tag{24}$$

For each $s \in \mathcal{S}$, let $x = Z$, $k = |\kappa(s)|$ and (with a slight abuse of notation) $w_i = d_i$, $i \in \kappa(s)$ to specialize the ILP (18) to

$$\zeta_s(Z) \equiv \max_{z_i: i\in\kappa(s)} \left\{ \sum_{i\in\kappa(s)} c_i z_i \middle| \sum_{i\in\kappa(s)} z_i \le Z, 0 \le z_i \le d_i, i \in \kappa(s) \right\}. \tag{25}$$

Since $\{\kappa(s), s \in \mathcal{S}\}$ is a partition of $\{1, \ldots, m\}$, following Equations (20)–(21) and observing that $\mathbf{z}^a$ and $\mathbf{z}^b$ are optimal solutions of $\phi(\mathbf{y}^a, \mathbf{d})$ and $\phi(\mathbf{y}^b, \mathbf{d})$ respectively,

$$\phi(\mathbf{y}^a, \mathbf{d}) = \sum_{s\in\mathcal{S}} \zeta_s(Z_s^a) \text{ and } \phi(\mathbf{y}^b, \mathbf{d}) = \sum_{s\in\mathcal{S}} \zeta_s(Z_s^b), \ s \in \mathcal{S}. \tag{26}$$

Let $x_1 = Z_s^a \wedge Z_s^b$, $x_2 = Z_s'$, $x_3 = Z_s''$, and $x_4 = Z_s^a \vee Z_s^b$. Then (23) and (24) match conditions for (19), so

$$\zeta_s(Z_s^a) + \zeta_s(Z_s^b) \le \zeta_s(Z_s') + \zeta_s(Z_s''), s \in \mathcal{S}. \tag{27}$$

Let $\{z_i', i \in \kappa(s)\}$ and $\{z_i'', i \in \kappa(s)\}$ be optimal solutions of $\zeta_s(Z_s')$ and $\zeta_s(Z_s'')$ respectively. Then

$$\sum_{s\in\mathcal{S}} [\zeta_s(Z_s') + \zeta_s(Z_s'')] = \sum_{s\in\mathcal{S}} \sum_{i\in\kappa(s)} c_i(z_i' + z_i'') = \sum_{i=1}^{m} c_i(z_i' + z_i'').$$

So by (26) and (27), (14) holds if $(z_1', \ldots, z_m')$ and $(z_1'', \ldots, z_m'')$ are feasible solutions of $\phi((\mathbf{y}^a - \alpha\mathbf{1}) \vee \mathbf{y}^b; \mathbf{d})$ and $\phi(\mathbf{y}^a \wedge (\mathbf{y}^b + \alpha\mathbf{1}); \mathbf{d})$ respectively.

Since $0 \le z_i \le d_i$ is a constraint of $\zeta_s(Z)$ $(s \in \mathcal{S})$,

$$0 \le z_i' \le d_i \text{ and } 0 \le z_i'' \le d_i, \quad 1 \le i \le m, \tag{28}$$

so what is left to be proved is that

$$\sum_{i=1}^{m} a_{ji} z_i' \le (y_j^a - \alpha) \vee y_j^b \text{ and}$$

$$\sum_{i=1}^{m} a_{ji} z_i'' \le y_j^a \wedge (y_j^b + \alpha), 1 \le j \le n, \quad (\text{where } a_{ji} \in \{0, 1\}),$$

which we do next.

Let s_j^0 be the smallest element of $\mathcal{S}$ that contains component j. Property (12) of the chained BOM implies that component j is in and only in sets that are elements of the chain

$$\mathbb{C}(s_j^0) = \{s_j^0, s_j^1, \ldots, s_j^k\}, \text{ where } s_j^l = p(s_j^{l-1}),\quad l = 1, \ldots, k \text{ and } s_j^k = \bar{s}.$$

Since $\{z_i', i \in \kappa(s)\}$ is an optimal (and thus feasible) solution of $\zeta_s(Z_s')$,

$$\sum_{i\in\kappa(s_j^0)} z_i' \le Z'_{s_j^0} = (\bar{Z}^a_{s_j^0} - \alpha) \vee \bar{Z}^b_{s_j^0} - (\bar{Z}^a_{s_j^1} - \alpha) \vee \bar{Z}^b_{s_j^1},$$

$$\sum_{i\in\kappa(s_j^1)} z_i' \le Z'_{s_j^1} = (\bar{Z}^a_{s_j^1} - \alpha) \vee \bar{Z}^b_{s_j^1} - (\bar{Z}^a_{s_j^2} - \alpha) \vee \bar{Z}^b_{s_j^2},$$

$$\cdots\cdots$$

$$\sum_{i\in\kappa(s_j^l)} z_i' \le Z'_{s_j^l} = (\bar{Z}^a_{s_j^l} - \alpha) \vee \bar{Z}^b_{s_j^l} - (\bar{Z}^a_{s_j^{l+1}} - \alpha) \vee \bar{Z}^b_{s_j^{l+1}},\ (l<k)$$

$$\cdots\cdots$$

$$\sum_{i\in\kappa(s_j^k)} z_i' \le Z'_{s_j^k} = (\bar{Z}^a_{\bar{s}} - \alpha) \vee \bar{Z}_{\bar{s}}.$$

Since $\kappa(s_j^l)$ ($l = 1, \ldots, k$) is a partition of $\mathcal{K}(s_j^0)$, by summing up the above equations,

$$\sum_{i\in\mathcal{K}(s_j^0)} z_i' \le (\bar{Z}^a_{s_j^0} - \alpha) \vee \bar{Z}^b_{s_j^0}. \tag{29}$$

Since all products that use component j (i.e., those with $a_{ji} = 1$) are in $\mathcal{K}(s_j^0)$,

$$\bar{Z}^a_{s_j^0} = \sum_{i\in\mathcal{K}(s_j^0)} z_i^a = \sum_{i=1}^m a_{ji} z_i^a \le y_j^a \text{ and}$$

$$\bar{Z}^b_{s_j^0} = \sum_{i\in\mathcal{K}(s_j^0)} z_i^b = \sum_{i=1}^m a_{ji} z_i^b \le y_j^b,$$

where the inequalities apply because $\mathbf{z}^a$ and $\mathbf{z}^b$ are optimal (and thus feasible) solutions of $\phi(\mathbf{y}^a, \mathbf{d})$ and $\phi(\mathbf{y}^b, \mathbf{d})$ respectively. Therefore

$$(\bar{Z}^a_{s_j^0} - \alpha) \vee \bar{Z}^b_{s_j^0} \le (y_j^a - \alpha) \vee y_j^b,$$

and thus by (29),

$$\sum_{i=1}^m a_{ji} z_i' = \sum_{i\in\mathcal{K}(s_j^0)} z_i' \le (y_j^a - \alpha) \vee y_j^b.$$

Since the above applies to any $j = 1, \ldots, n$, $(z_1', \ldots, z_m')$ is a feasible solution of $\phi((\mathbf{y}^a - \alpha\mathbf{1}) \vee \mathbf{y}^b, \mathbf{d})$.

Following a similar procedure, we can also prove that

$$\sum_{i=1}^m a_{ji} z_i'' \le y_j^a \wedge (y_j^b + \alpha),\quad j = 1, \ldots, n,$$

so $(z_1'', \ldots, z_m'')$ is a feasible solution of $\phi(\mathbf{y}^a \wedge (\mathbf{y}^b + \alpha\mathbf{1}), \mathbf{d})$. □

Remark 3. Chen et al. (2016) studies a multi-period ATO system with random capacities. They consider the generalized M system where there are $n + 1$ products and n component, product 0 uses one unit of all components while product i uses one unit of componet i ($i = 1, \ldots, n$). As a special case of Theorem 5 in Chen et al. (2016) (limiting the number of periods to 1), the cost function of the system is $L^\natural$ convex. Our theorem also implies this result as a special case.

Observe that by changing $\mathbb{Z}^k_+$ to $\mathbb{Z}^k$ in (18) and removing $z_i \ge 0$ in (25) and around (28), the same proof applies to the relaxed SP (4). Thus we conclude

Theorem 2. *In any system with a chained BOM,*

$$\varphi(\mathbf{y}^a; \mathbf{d}) + \varphi(\mathbf{y}^b; \mathbf{d}) \le \varphi((\mathbf{y}^a - \alpha\mathbf{1}) \vee \mathbf{y}^b; \mathbf{d}) + \varphi(\mathbf{y}^a \wedge (\mathbf{y}^b + \alpha\mathbf{1}); \mathbf{d}) \tag{30}$$

for any values of $\mathbf{y}^a, \mathbf{y}^b \in \mathbb{Z}^n$, $\alpha \in \mathbb{Z}_+$, and $\mathbf{d} \in \mathbb{Z}^m_+$. Thus $\hat{C}(\mathbf{y})$ is $L^\natural$ convex everywhere and under any demand distribution.

3.3. Discussion on Other ATO Systems

Having established that the chained BOM structure is a sufficient condition for $L^\natural$ convexity, we now consider whether the property holds for other structures. There can be two possible deviations: (i) the BOM matrix has non-binary entries so products can use multiple units of a component; and (ii) condition (12) does not hold, so different products are built from component sets that only partially overlap with each other. In each case, there are specific parameter values and demand distributions that can preserve $L^\natural$ convexity. Nevertheless, our examples below show that in general, the property cannot hold in common situations under each extension.

Our examples apply to the one-period ATO model discussed in Theorem 1. We focus on situations where (14) is violated, thus $\phi(\mathbf{y}; \mathbf{d})$ is not $L^\natural$ concave, for some values of $\mathbf{d}$: $L^\natural$ convexity of $C(\mathbf{y})$ cannot survive these cases for demand distributions that are heavily concentrated on these values.

Example 1. As a simple case where entries of A can be any value in $\mathbb{Z}_+$, consider an ATO system with only one component. We can drop the component index and denote by a_i the number of units of the component used by product i ($1 \le i \le m$). The one-period ATO model (6) specializes to

$$C(\mathbf{y}) \equiv \sum_{i=1}^{m} b_i E[D_i] + hy - E[\phi(y;\mathbf{D})]$$
$$\text{where } \phi(y;\mathbf{d}) = \max_{\mathbf{z}} \left\{ \sum_{i=1}^{m} c_i z_i \middle| \sum_{i=1}^{m} a_i z_i \le y, \; 0 \le z_i \le d_i, i = 1, \ldots, m \right\}.$$

and $L^\natural$ concavity amounts to discrete concavity.

In situations where $\mathbf{d} = (1, \ldots, 1)$, $\phi(y; \mathbf{d})$ specializes to the following Knapsack problem

$$\phi(y;\mathbf{1}) = \max_{\mathbf{z}} \left\{ \sum_{i=1}^{m} c_i z_i \middle| a_1 z_1 + \cdots + a_m z_m \le y, z_i \in \{0,1\}, \; i = 1, \ldots, m \right\}.$$

This function is normally not discretely concave in y. For instance, without loss of generality, assume

$$\frac{c_1}{a_1} \ge \frac{c_2}{a_2} \ge \cdots \ge \frac{c_m}{a_m}. \quad (31)$$

Let product $k = \min\{i : a_i > 1, 1 \le i \le m\}$ be the one that has the largest c/a ratio among those using more than one unit of the component. With a slight loss of generality, assume

$$k < m \text{ and } \frac{c_k}{a_k} > \frac{c_{k+1}}{a_{k+1}}.$$

Let $y^a = k - 1 + a_k$, $y^b = k - 1$, and $\alpha = 1$. Then (by definition, $a_1 = \cdots = a_{k-1} = 1$),

$$\phi(y^a;\mathbf{1}) \ge c_1 + \cdots + c_{k-1} + c_k$$

because $z_1 = \cdots = z_k = 1$ and $z_{k+1} = \cdots = z_m = 0$ is a feasible solution. For a similar reason,

$$\phi(y^b;\mathbf{1}) \ge c_1 + \cdots + c_{k-1}.$$

On the other hand, since $(y^a - 1) \vee y^b = k - 1 + a_k - 1$ and because of (31),

$$\phi((y^a - 1) \vee y^b;\mathbf{1}) \le c_1 + \cdots + c_{k-1} + \frac{c_k}{a_k}(a_k - 1) = c_1 + \cdots + c_k - \frac{c_k}{a_k},$$

and since $y^a \wedge (y^b + 1) = k$, $a_k > 1$, and $c_{k+1}/a_{k+1} < c_k/a_k$,

$$\phi(y^a \wedge (y^b + 1);\mathbf{1}) < c_1 + \cdots + c_{k-1} + \frac{c_k}{a_k},$$

and (14) does not hold as a consequence.

Example 2. As a simple case of violating (12), consider the W system shown in Figure 1. Products 1 and 2 use a common component 0 (slight deviation from the standard notation). Each also uses a separate component, 1 and 2 respectively. The one-period ATO system (6) becomes

$$C(\mathbf{y}) \equiv \sum_{i=1}^{2} b_i E[D_i] + \sum_{j=0}^{2} h_j y_j - E[\phi(\mathbf{y};\mathbf{D})]$$
$$\text{where } \phi(\mathbf{y};\mathbf{d}) = \max_{\mathbf{z} \ge \mathbf{0}} \{c_1 z_1 + c_2 z_2 | z_1 \le d_1, z_2 \le d_2, \; z_1 \le y_1, z_2 \le y_2, z_1 + z_2 \le y_0\}.$$

Assume that $c_1 \ge c_2$. It is easy to check that if $y_1 \le d_1$, $y_2 \le d_2$, and $y_1 \le y_0$,

$$\phi(\mathbf{y};\mathbf{d}) = c_1 y_1 + c_2[(y_0 - y_1) \wedge y_2].$$

Apply this solution to any given $d_1 > 0$, $d_2 > 0$, and values of $\mathbf{y}^a$, $\mathbf{y}^b \in \mathbb{Z}_+^2$ that satisfy

$$\begin{aligned} & y_1^a \vee y_1^b \le d_1, y_2^a \vee y_2^b \le d_2, \\ & y_1^a \le y_0^a \le y_1^a + y_2^a, y_1^b \le y_0^b \le y_1^b + y_2^b, \\ \text{and } & y_0^a \wedge y_0^b > y_1^a \wedge y_1^b + y_2^a \wedge y_2^b. \end{aligned} \quad (32)$$

We then arrive at

$$\phi(\mathbf{y}^a;\mathbf{d}) + \phi(\mathbf{y}^b;\mathbf{d}) = c_1(y_1^a + y_1^b) + c_2(y_0^a - y_1^a + y_0^b - y_1^b),$$

so that (since (32) implies $y_1^a \vee y_1^b \le y_0^a \vee y_0^b$),

$$\begin{aligned} & \phi(\mathbf{y}^a \vee \mathbf{y}^b;\mathbf{d}) = c_1(y_1^a \vee y_1^b) + c_2(y_0^a \vee y_0^b - y_1^a \vee y_1^b), \\ \text{and } & \phi(\mathbf{y}^a \wedge \mathbf{y}^b;\mathbf{d}) < c_1(y_1^a \wedge y_1^b) + c_2(y_0^a \wedge y_0^b - y_1^a \wedge y_1^b), \end{aligned}$$

where the second (strict) inequality results from the last inequality in (32). Therefore

$$\phi(\mathbf{y}^a \wedge \mathbf{y}^b;\mathbf{d}) + \phi(\mathbf{y}^a \vee \mathbf{y}^b;\mathbf{d}) < \phi(\mathbf{y}^a;\mathbf{d}) + \phi(\mathbf{y}^b;\mathbf{d}),$$

which violates (14). For instance, when $(y_0^a, y_1^a, y_2^a) = (6, 4, 3)$ and $(y_0^b, y_1^b, y_2^b) = (7, 2, 6)$, $d_1 \ge 4$, and $d_2 \ge 6$,

$$\begin{aligned} & \phi(\mathbf{y}^a;\mathbf{d}) + \phi(\mathbf{y}^b;\mathbf{d}) - \phi(\mathbf{y}^a \wedge \mathbf{y}^b;\mathbf{d}) - \phi(\mathbf{y}^a \vee \mathbf{y}^b;\mathbf{d}) \\ & = (4c_1 + 2c_2) + (2c_1 + 5c_2) - (2c_1 + 3c_2) - \\ & (4c_1 + 3c_2) > 0. \end{aligned}$$

Remark 4. As mentioned earlier, the W system is in the tree family defined in Zipkin (2016). It is easy to see that having $y_0 > y_1 + y_2$, as the last inequality in (32), is never optimal. Thus, as is shown by Zipkin (2016), while the SP is not $L^\natural$ convex, one can impose additional restrictions on $\mathbf{y}$ without loss of generality. Consequently, the problem is cover-$L^\natural$-convex, a weaker property than $L^\natural$ convexity, but still admits an efficient solution procedure.

However, it is easy to see that the W structure can be embedded in more general ATO systems that are not in the tree family. With many more components and products, excluding $y_0 > y_1 + y_2$ compromises optimality, in which case neither $L^\natural$ convexity nor cover-$L^\natural$-convexity applies.

4. SP-Based Inventory Control for the M system

As alluded to in section 1, to evaluate policy performance and characterize possible allocation solutions, we apply the SP-based approach to the M system. In section 4.1, we provide explicit expressions for the objective functions of both SPs (4) and (6) to simplify the calculation of the lower bound and base stock levels. In section 4.2, we transform the allocation policy into simple rules that can be easily implemented. These rules are qualitatively different for different parameter regions.

4.1. SP Solutions: Lower Bound and Replenishment Policy

The M system in Figure 1 obviously has a chained BOM structure. Thus

COROLLARY 1. *In the M system, $C(\mathbf{y})$ and $\hat{C}(\mathbf{y})$ are $L^\natural$ convex.*

The corollary paves the way for using a steepest decent algorithm to search for the optimal solution (Murota 2003b). Still, we need to decide how to evaluate $C(\mathbf{y})$ and $\hat{C}(\mathbf{y})$ economically at each search step. Generic numerical procedures, such as sample average approximation (SAA), are available for the task. Nevertheless, these methods usually involve a trade-off between the accuracy of the solution and calculation burden (Shapiro and Nemirovsky 2005). Thus it is worthwhile to, as we do next, develop an explicit expression of $C(\mathbf{y})$ and $\hat{C}(\mathbf{y})$ based on the M system structure, so these values can be calculated directly.

From (4) and (6), both $C(\mathbf{y})$ and $\hat{C}(\mathbf{y})$ can be commonly expressed as

$$\sum_{i=0}^{2} b_i E[D_i] + \sum_{j=1}^{2} h_j y_j - \sum_{i=0}^{2} c_i E[z_i^*(\mathbf{y};\mathbf{D})],$$

where $z_i^*(\mathbf{y};\mathbf{D})(i = 0, 1, 2)$ is an optimal solution to the recourse ILP in the associated SP. The explicit form for each of these solutions is given by the following lemma, which we prove in Appendix A.1.

LEMMA 1. *In either (5) or (7), if (z_0^*, z_1^*, z_2^*) optimizes the second-stage recourse problem for given $\mathbf{D}$, then*

$$z_1^* = D_1 \wedge (y_1 - z_0^*) \text{ and } z_2^* = D_2 \wedge (y_2 - z_0^*). \quad (33)$$

Let $\tilde{y}_1 = y_1 - D_1, \tilde{y}_2 = y_2 - D_2$, values of z_0^* in different parameter regions are

SP	$c_1 + c_2 < c_0$	$c_2 \le c_1 < c_0 \le c_1 + c_2$	$c_2 < c_0 \le c_1$	$c_0 \le c_2 \le c_1$
(5)	D_0	$D_0 \wedge (\tilde{y}_1 \vee \tilde{y}_2)$	$D_0 \wedge \tilde{y}_1$	$D_0 \wedge \tilde{y}_1 \wedge \tilde{y}_2$
(7)	$D_0 \wedge y_1 \wedge y_2$	$D_0 \wedge (\tilde{y}_1^+ \vee \tilde{y}_2^+) \wedge y_1 \wedge y_2$	$D_0 \wedge \tilde{y}_1^+ \wedge y_2$	$D_0 \wedge \tilde{y}_1^+ \wedge \tilde{y}_2^+$

Using the lemma, we derive $E[\mathbf{z}^*(\mathbf{y};\mathbf{D})]$ as explicit functions of $\mathbf{y}$, and this allows direct evaluation of $C(\mathbf{y})$ and $\hat{C}(\mathbf{y})$ without sampling. For instance, when $c_0 > c_1 + c_2$, by the expressions in the second column of the table, for SP (6), the one-period ATO model,

$$\begin{aligned} E[z_0^*] &= E[D_0 \wedge y_1 \wedge y_2], \\ E[z_1^*] &= \sum_{k_0=0}^{y_1 \wedge y_2} f_0(k_0) E[D_1 \wedge (y_1 - k_0)] + \bar{F}_0(y_1 \wedge y_1) \\ &\qquad E[D_1 \wedge (y_1 - y_2)^+], \\ E[z_2^*] &= \sum_{k_0=0}^{y_1 \wedge y_2} f_0(k_0) E[D_2 \wedge (y_2 - k_0)] + \bar{F}_0(y_1 \wedge y_2) \\ &\qquad E[D_2 \wedge (y_2 - y_1)^+], \end{aligned}$$

where $f_i(x)$ and $F_i(x)$ are probability density and cumulative probability functions of demand D_i (i = 0, 1, 2). For the relaxed SP (4),

$$E[z_0^*] = E[D_0] \text{ and } E[z_i^*] = \sum_{k_0=0}^{\infty} f_0(k_0) E[D_i \wedge (y_i - k_0)] \quad i = 1, 2.$$

Similar expressions for other parameter regions are derived in Appendix A.2. These developments, combined with the proof of $L^\natural$ convexity, reduce both SPs to single-stage optimization problems for which the optimal solutions can be found directly by searching for a local minimum.

4.2. The Allocation Policy

For the M system, component shortages in (9) specialize to

$$Q_j(t) = B_j(t) + B_0(t) - I_j(t), \quad j = 1, 2. \quad (34)$$

Correspondingly, the ILP (8) for determining the backlog targets specialize to

$$\begin{aligned} \mathbf{B}^* = \operatorname{argmin}_{\mathbf{B} \ge 0} \{ & c_0 B_0 + c_1 B_1 + c_2 B_2 | B_j \\ & + B_0 \ge Q_j, \ j = 1, 2 \}. \end{aligned} \quad (35)$$

Below we discuss policies that implement the aforementioned "Allocation Principle" in each of the four different parameter regions.

4.2.1. Region A ($c_1 + c_2 < c_0$): In this parameter region, the optimal solution of (35) is

$$B_0^* = 0, B_1^* = Q_1^+, B_2^* = Q_2^+, \tag{36}$$

i.e., the backlog target for product 0 is always zero and any component shortage should be allocated to the other two products. Applying the Allocation Principle with this solution yields the following policy: *serve demand of product 0 whenever possible and serve product i (i = 1, 2) if and only if the remaining amount of component i is sufficient to clear all existing backlog of product 0.*

The policy draws a sharp contrast with myopic/no holdback policies in Lu et al. (2010). The latter requires that no component can remain in inventory if it can be used to clear an additional unit of backlog, regardless of the product. In the M system, the no holdback condition can be expressed by

$$\begin{aligned} I_1(t) \wedge B_1(t) &= I_2(t) \wedge B_2(t) \\ &= I_1(t) \wedge I_2(t) \wedge B_0(t) = 0, \text{ for all } t, \end{aligned} \tag{37}$$

which is not satisfied by our SP-based policy. Consider for example, the case where

$$B_0^-(t) = 1,\ B_1^-(t) = 1,\ B_2^-(t) = 0,\ I_1^-(t) = 1,\ I_2^-(t) = 0.$$

At time t, both components have shortage of one unit ($Q_1(t) = Q_2(t) = 1$). No component 2 is available, so neither product 0 nor product 2 can be served. There is one unit of component 1, which will be used under a myopic/no holdback policy to serve the existing demand of product 1. Under our policy, this last unit will remain in inventory and product 1 can be served only after the system has reserved a component 1 for every unit of existing backlog of product 0. Similarly, when component 1 is not available, the policy reserves component 2 for all backlogged product 0.

To explain the need for reservation, consider the situation when both components have deficits and all three products have outstanding backlogs. If no component is reserved (held back), then product 0, despite its importance, is served only when replenishments of both components arrive. By holding back whichever component that is currently available, product 0 can use replenishments of both components, as well as replenishments of the missing item, so its backlog is cleared more quickly. While reservation idles usable components, the additional inventory cost from idling can be small in comparison with savings from serving the important product faster, especially in an asymptotic region where both demands and replenishments arrive at fast rates, so the idling does not last long.

4.2.2. Region B ($c_2 \le c_1 < c_0 \le c_1 + c_2$): A manufacturer may sell two individual products both separately and in a bundle. To maximize the profit, the bundle price can be set lower than the sum of prices of the two separate products, but higher than the price of each individual item. If the backlog cost corresponds to the delayed revenue, then $b_1, b_2 < b_0 \le b_1 + b_2$, implying the cost region of $c_2 \le c_1 < c_0 \le c_1 + c_2$, When the value of serving products 0 is higher than serving either product 1 or 2 but lower than serving them both,

$$B_0^* = Q_1^+ \wedge Q_2^+, B_1^* = (Q_1 - Q_2^+)^+, B_2^* = (Q_2 - Q_1^+)^+. \tag{38}$$

The solution indicates that when only component 1 has shortage ($Q_1(t) > 0,\ Q_2(t) \le 0$), both products 0 and 2 should have no backlog, which is feasible for product 2 since there is no lack of component 2. For product 0, the Allocation Principle dictates that it has the priority to use all component 1 until after its backlog is completely cleared, at which point product 1 can be served. Similarly, when only component 2 has shortage, all backlogs of product 1 should be cleared, and product 0 should be given priority of using component 2 over product 2.

When both components have shortages, ($Q_1(t) > 0$, $Q_2(t) > 0$), then the backlog target of product 0 is the minimum of the two shortage levels. The difference between the shortage level of component i from the minimum level sets the backlog target of product i (i = 1, 2), so one of these targets is 0. Correspondingly, when all three products have outstanding backlogs, all components are used to serve products 1 and 2, until one of them has no existing demand. At this point, product 0 gets the priority and the other product that still has existing demand is served if and only if the component it needs cannot be used immediately by product 0.

A state-dependent priority rule summarizes the above situations: *the higher priority is given to products 1 and 2 when they both have existing demands, and to product 0 if one of the other two products has no backlog*. The policy matches the cost value in this region, where the unit inventory cost of product 0 is higher than individual costs of product 1 and 2, but lower than their sum.

4.2.3. Region C ($c_2 < c_0 \le c_1$): In this case, product 1 consumes fewer components and is more valuable to serve than product 0, and

$$B_0^* = Q_1^+, B_1^* = 0, B_2^* = (Q_2 - Q_1^+)^+.$$

Applying the Allocation Principle with this solution leads to a static priority policy. The backlog target of product 1 is 0, so *component 1 should be used to serve its existing demand first*. The target of product 0 is the shortage level of component 1, so *product 0 has the priority to use component 2 as long as component 1 is still available after satisfying all demands of product 1*. The target of product 2 implies that *product 2 should be served at and only at times when component 2 is not needed by product 0*.

4.2.4. Region D ($c_0 \leq c_2 \leq c_1$): In this last case,

$$B_0^* = Q_1^+ \vee Q_2^+, B_1^* = B_2^* = 0.$$

Following the same discussion as in Region C: *both products 1 and 2 have the priority to use available components; product 0 is served only when no other product has backlog*.

4.2.5. Summary. In an ATO inventory system, the relative values of products are determined by their unit inventory costs. While demands of higher-value products should always be satisfied first, allocation policies that implement this principle may vary qualitatively according to the BOM structure. On the one hand, a myopic priority rule is sufficient in cases where higher-value products require fewer components than other products (Cases B, C and D). On the other hand, if a higher-value product uses many components that are used individually by lower-valued products (Case A), then the policy will reserve (hold back) components.

5. Numerical Studies

We evaluate the performance of our policy, referred to as SP, in an M system in which demands for different products arrive according to independent Poisson processes. As previously described, we use the stochastic program (6)–(7) to set base stock levels and follow the allocation policy described in section 4. Our evaluation is carried out by comparing SP with the following alternatives.

1. Our first benchmark is the policy in Lu and Song (2005), which we refer to as LS. By optimizing base stock levels, LS dominates all continuous-review policies that follow a base stock replenishment policy and the FIFO (with component commitment) allocation policy. The base stock levels and the value of the expected cost objective are both calculated based on the same method laid out in Lu and Song (2005).
2. We also compare our approach with the no-holdback (NHB) approach proposed in Lu et al. (2010). As mentioned earlier, NHB is an allocation principle that does not allow components to stay idle in inventory if they can be used to clear existing backlogs. Lu et al. (2010) implements the NHB principle by the FRFS policy, which is similar to LS except that no component is committed to demands that are not ready to be served due to the lack of other required components. Their paper applies FRFS under a base stock replenishment policy but comments in section 6 that optimizing the base stock levels for this policy is difficult. Since our purpose is to compare SP-based approach with existing alternatives and their variants, we continue to use the same base stock levels as in Lu and Song (2005).
3. FRFS is not typically the best implementation of the NHB principle. For the M system, a better NHB policy is the one that prioritizes the service to different products based on their unit inventory costs (c_i, i = 0, 1, 2), and thus minimizes the immediate inventory cost. For a comparison with SP, we define a new policy, referred to as LSP, that implements this allocation scheme and uses the method in Lu and Song (2005) to set base stock levels. In parameter regions B, C, and D, LSP coincides with SP in allocation policy. In region A, LSP is similar to SP by giving priority to product 0, but differs from it by not holding back components for the product. In all cases, the base stock levels are different between LSP and SP.

We use the percentage gap between the inventory cost of a policy and the lower bound, $\hat{C}^*$, as the performance measure of our comparisons. Specifically, the percentage gap is defined as

$$\Delta_p = 100 \frac{C^p - \hat{C}^*}{\hat{C}^*}$$

Here C^p is the long-run average inventory cost under policy p where p can be LS, FRFS, LSP, or our policy (SP). The value of C^p is estimated by simulation or (in the case of LS) calculated as an explicit solution. A small percentage gap is an indication of policy p performing close to optimal. Even though a large gap does not provide enough information regarding optimality (since the lower bound is typically not attainable), it supplies a basis for performance comparison of different policies.

5.1. Asymptotic Optimality

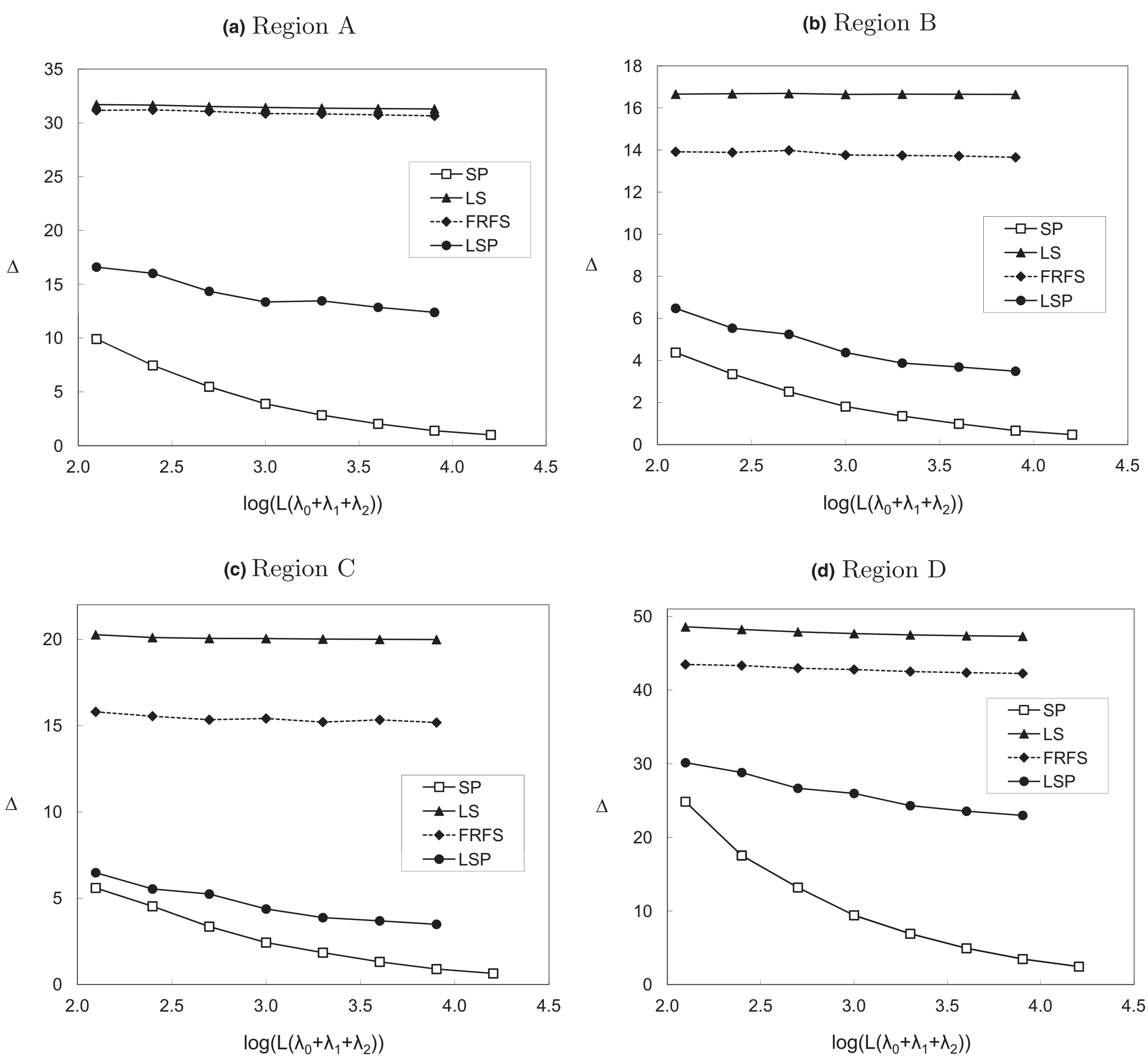
We first compare the asymptotic behavior of different policies. We demonstrate asymptotic optimality of SP, which is a proven result (Reiman and Wang 2015). We also provide numerical evidence that, coupled with analogous arguments in previous work, leads us to conjecture that alternative policies are not asymptotically optimal. Our discussion reveals that both the values of base stock levels and the allocation of components can affect asymptotic optimality.

For brevity, we focus on a particular parameter region. We set component holding costs at $h_1 = h_2 = 1$ and demand arrival rates at $\lambda_0 = 25$ and $\lambda_1 = \lambda_2 = 50$. We create one scenario for each of the four aforementioned cases by setting backlog costs at

$$\begin{aligned}\text{Region A: } & b_0 = 8, b_1 = 3.5,\ b_2 = 1, \\ & (c_0 = 10, c_1 = 4.5, c_2 = 2), \\ \text{Region B: } & b_0 = 3, b_1 = 2.5, b_2 = 1, \\ & (c_0 = 5, c_1 = 3.5, c_2 = 2), \\ \text{Region C: } & b_0 = 2, b_1 = 3.5, b_2 = 1, \\ & (c_0 = 4, c_1 = 4.5, c_2 = 2), \\ \text{Region D: } & b_0 = 1, b_1 = 8, b_2 = 3, \\ & (c_0 = 3, c_1 = 9, c_2 = 4).\end{aligned}$$

We vary the lead time from $L = 1$ to $L = 128$ and examine the corresponding optimality gap (Δ). The results are shown in Figure 2 where the increase of the lead time is shown at the log scale (the method in

Figure 2 Asymptotic Optimality of Different Policies



Lu and Song (2005) for calculating the optimal base stock level does not scale well with the lead time, so Figure 2 does not include data points of LS, FRFS, and LSP for $L = 128$). As can be observed from the figure:

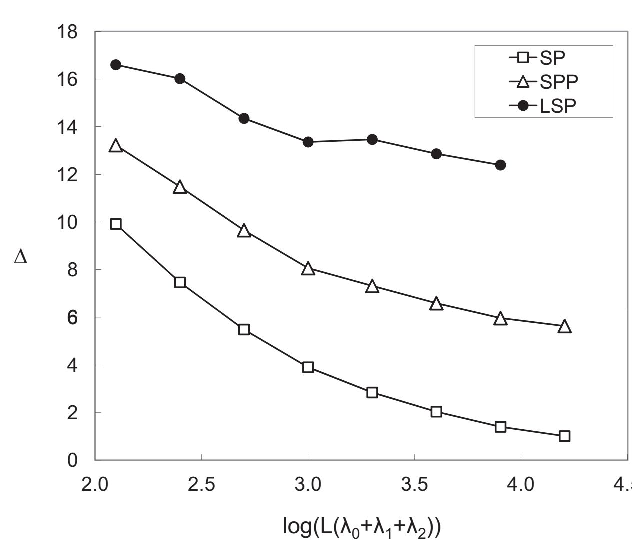
1. In all four cases, the optimality gap (Δ) of SP converges to 0 as L grows, which is expected from the proof of asymptotic optimality in Reiman and Wang (2015).
2. A strictly positive optimality gap (Δ) persists in all four cases under two FIFO-based policies, LS and FRFS, suggesting neither one is asymptotically optimal. This is not surprising in light of the discussion in Reiman and Wang (2015, p. 724) on the inverse-V system, which has only one component that is used to assemble two products with different backlog costs. They argue that asymptotic optimality cannot be attained without serving the higher-value product with priority. Since the inverse-V system is a special case of many ATO systems, including the M system (by removing either product and component i, $i = 1, 2$), their conclusion applies here, and is manifested by numerical results.
3. Under the same base stock levels but with a different implementation of the NHB principle, LSP significantly outperforms FRFS. However, the improvement is not sufficient to make the optimality gap disappear as the lead time grows. The apparent failure of LSP to achieve asymptotic optimality in region A can be explained by the fact that unlike SP, the policy does not reserve components. Wan and Wang (2015) proves that in any related ATO inventory-production system that embeds an M structure with $c_0 > c_1 + c_2$, no allocation policy can be asymptotically optimal without reserving components for product 0. They consider the high-volume asymptotic regime that is equivalent to the long lead time regime when the lead times are identical. Therefore, with minor twists, their argument can be applied analogously here to support our observation from the numerical results that LSP is not asymptotically optimal.
4. Besides the difference in allocation, LSP also implements different base stock levels from SP. To assess the impact of this difference on performance, we consider a mixed policy, SPP, which follows the same base stock levels as SP and the same allocation policy as LSP. Figure 3 shows a comparison of LSP, SPP, and SP, for the same cases as those in Figure 2, Region A. As is expected, without reservation, the optimality gap of SPP also appears to fail to converge to 0 as the lead time grows. Still, its optimality gap is significantly smaller than that of LSP, highlighting the impact of base stock levels on asymptotic optimality. It is reasonable to conjecture that if FIFO allocation is not asymptotically optimal, the base stock levels of LSP, which is optimized for FIFO allocation, will have an optimality gap even though the actual allocation is more efficient. The conjecture is consistent with further comparisons in the rest of Figure 2, between LSP and SP in regions B, C, and D. Allocation policies coincide and the two policies differ only in base stock levels, and the difference is sufficient to lead to a divergence in optimality gaps.

Observe that the magnitude of the above asymptotic divergence is non-trivial. As Figures 2 and 3 show, while the optimality gap of SP converges to zero, that of its nearest competitor hovers around 3% to more than 20%. In a billion-dollar inventory system, this positive gap represents millions if not tens of millions of dollars of additional annual inventory expenses.

5.2. Performance Comparison in Non-Asymptotic Regimes

While a proof of asymptotic optimality does not guarantee that SP is superior in all cases, it suggests that the policy may have some desirable properties that can yield performance advantages even outside the asymptotic regime. Our numerical studies below discuss to what extent SP can dominate other approaches in cases where the lead time is not long enough to allow the asymptotic behavior to prevail, that is, where SP still has a non-trivial optimality gap.

Figure 3 Component Reservation and Asymptotic Optimality (Region A)



The testbed for these comparisons is created as follows: we have assumed without the loss of generality that $c_2 \le c_1$, which implies that $b_2 - h_1 \le b_1 - h_2$ and thus

$$b_2 - h_1 \le b_1 - h_2 < b_1 + b_2.$$

Correspondingly, the aforementioned four parameter regions can be characterized by

$$\text{Region A } (c_1 + c_2 < c_0) : b_1 + b_2 < b_0,$$
$$\text{Region B } (c_1 < c_0 \le c_1 + c_2) : b_1 - h_2 < b_0 \le b_1 + b_2,$$
$$\text{Region C } (c_2 < c_0 \le c_1) : b_2 - h_1 < b_0 \le b_1 - h_2,$$
$$\text{Region D } (c_0 \le c_2) : b_0 \le b_2 - h_1.$$

For regions A and D, we set b_0 to be 30% away from its boundary values, that is,

$$b_0 = 1.3(b_1 + b_2) \text{ and}$$
$$b_0 = 0.7(b_2 - h_1) \text{ for regions A and D respectively.}$$

For regions B and C, we set b_0 at the mid-point of the corresponding interval, that is,

$$b_0 = b_1 + \frac{1}{2}(b_2 - h_2) \text{ and } b_0 = \frac{1}{2}(b_1 + b_2 - h_1 - h_2)$$

for regions B and C, respectively.

To set other parameter values, without loss of generality, we fix $h_2 = 1$. We consider cases in which h_1 is e_1% of h_2 where $e_1 = 50, 100, 150$, and cases in which b_2 is e_2% of h_2 where $e_2 = 60, 160, 250$. To specify the value of b_1, we consider cases in which $c_1 = 1.5c_2$ and $c_1 = 2c_2$. Eliminating cases where $b_2 \le h_1$ (in which case b_0 would be negative in region D), the above specifications result in 56 possible combinations of $(b_0, b_1, b_2, h_1, h_2)$. For each of these combinations, we let demand arrival rates be $(\lambda_0, \lambda_1, \lambda_2) = (20, 20, 20)$, $(10, 20, 20)$, $(20, 10, 20)$ and $(20, 20, 10)$ to generate 224 test cases for our numerical study. The lead time is fixed at $L = 1$.

For these test cases, Table 1 compares the optimality gap of SP with those of LS, FRFS, and LSP. Rows 2–6 show numbers of cases in which SP outperforms LS, FRFS, and LSP in various degrees. The next three rows show the numbers of cases in which SP underperforms these alternatives. For instance, in 60 of these 224 comparisons,

$$\Delta_{LS} - \Delta_{SP} > 20\%,$$

so in the table, the number of cases in which SP is overwhelmingly better than LS is 60. As another example, the table shows that in 6 cases, SP is somewhat worse than LSP, which means that in 6 out of 224 comparisons, Δ_{SP} exceeds Δ_{LSP} by an amount between 3% and 10%.

The table shows that the optimality gap of SP is strictly lower than those of LS and FRFS in all 224 cases, and significantly so in a majority of them. These results demonstrate that even outside the asymptotic regime, SP has an indisputable advantage over the two common approaches in the literature. SP also outperforms LSP in 191 out of 224 cases, but in about 60% of these cases, the difference of the optimality gap is <3%.

To put these results into perspective, LS, FRFS, and LSP all follow the same replenishment policy, so their performance differences are attributed entirely to component allocation. FIFO (implemented by LS) and FRFS allocations are thoroughly compared in Lu et al. (2010) where the advantage of the latter is discussed in detail. Particularly for the M system, Theorem 5 in Lu et al. (2010) shows that FRFS results in a lower level of total backlogs on every sample path, which helps to reduce the inventory cost. However, the same theorem also shows that by not holding back components, FRFS leads to a higher backlog level of product 0, which can make the situation worse if product 0 has a much higher inventory cost than the other two products. LSP retains the better feature of FRFS by not holding components for low-value products, and mitigates its deficiency by allocating components based on products' inventory cost instead of their arrival sequence. Hence, it is not surprising to see that the policy is a distinct front runner among the three, well ahead of LS and FRFS. SP makes additional improvements by withholding components for product 0 if and only if its inventory cost is sufficiently higher than those of other products. So it is also not a surprise to see that it surpasses LSP in many cases.

Table 1 Comparison of the Optimality Gap between Stochastic Programs (SP) and Other Approaches

	Degree	$\Delta_p - \Delta_{SP}$	p = LS	p = FRFS	p = LSP
Better	Overwhelmingly	>20%	60	34	14
	Significantly	(10%, 20%]	66	56	12
	Somewhat	(3%, 10%]	97	128	45
	Slightly	(0.5%, 3%]	1	6	97
	Barely visible	(0%, 0.5%]	0	0	23
Worse	Barely visible	(−0.5%, 0%]	0	0	11
	Slightly	(−3%, −0.5%]	0	0	16
	Somewhat	(−10%, −3%]	0	0	6

Of 224 comparisons between SP and LSP, 168 of them are based on test cases in regions B, C, and D, and SP performs better in 157 of them. Since the approaches follow the same priority allocation policy, better performance of SP is solely a result of better replenishment decisions. Specifically, in setting base stock levels, SP recognizes that unlike FIFO, priority allocation diverts components to more valuable products, so fewer components are needed to reduce the total backlog cost to the same level. This impact is most significant in region D where the priority policy dictates that product 0, which consumes both components, can be served only if neither product 1 nor product 2 has backlog. As an illustration, Figures 4a and b show base stock levels of the two policies. Those of LSP, represented by the dotted lines, are consistently higher than those of SP in all cases. The resulting difference in the optimality gap is shown in Figure 4c.

The most striking comparison in Figure 4 is for scenario 55 with the following parameter values

$$h_1 = 1.5,\ h_2 = 1,\ b_0 = 0.07,\ b_1 = 3.7,\ b_2 = 1.6,$$
$$\lambda_0 = \lambda_1 = 20,\ \text{and}\ \lambda_2 = 10.$$

While backlog costs of products 1 and 2 are higher than inventory holding costs of their required components, the backlog cost of product 0 is an order of magnitude smaller. Assuming that all component shortages will be absorbed by the backlog of product 0 under priority allocation, the base stock levels under SP are set at 32 (component 1) and 23 (component 2). These levels are below the average demands for components 1 ($\lambda_0 + \lambda_1 = 40$) and 2 ($\lambda_0 + \lambda_2 = 30$), so the system will have persistent deficits of component supply. Assuming FIFO allocation, which implies that all three products will have comparable backlogs, LSP sets much higher

Figure 4 Comparison of Stochastic Program (SP) and LSP in Region D

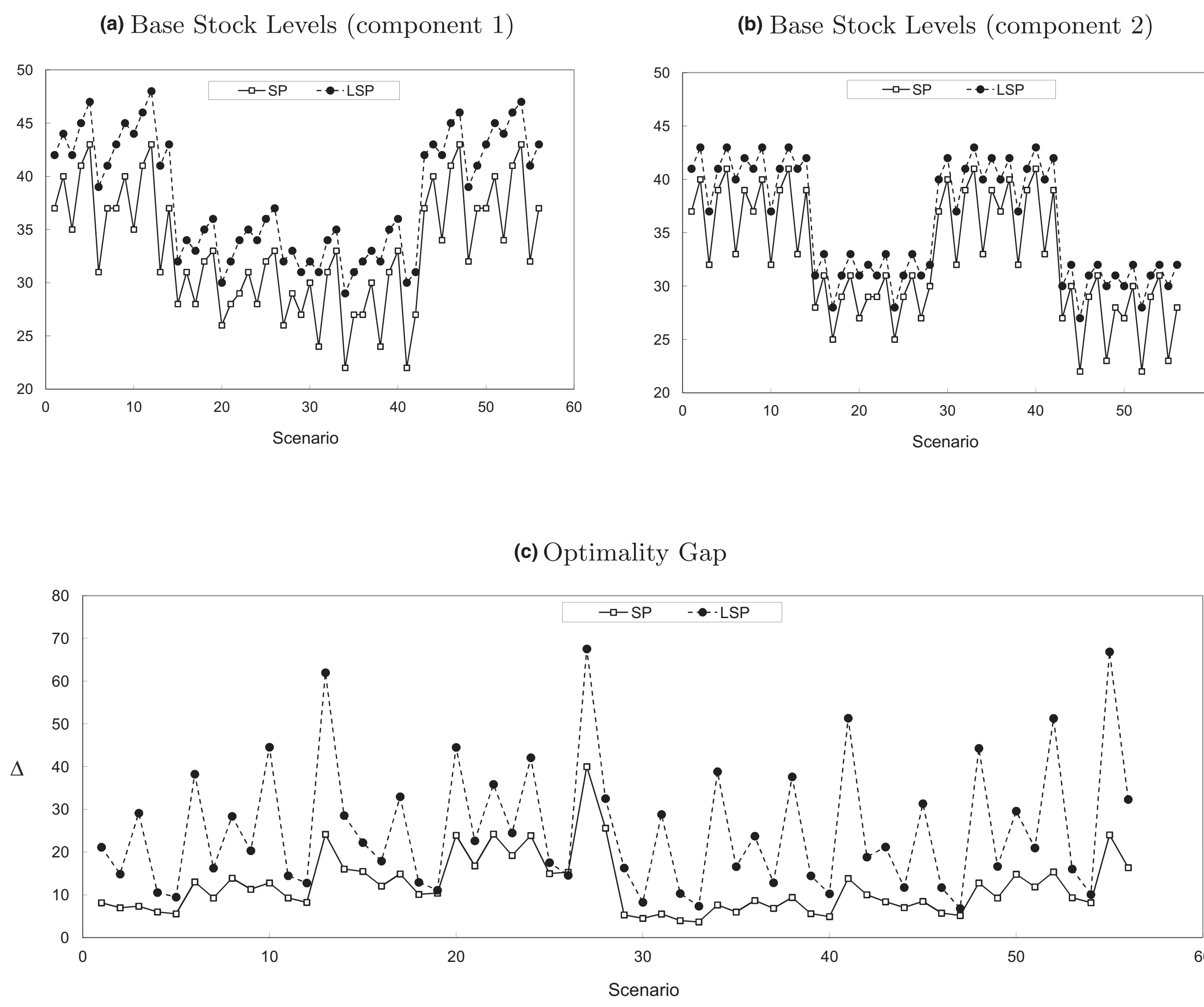


Table 2 Breakup of Inventory Costs: SP vs. LSP

	Holding cost		Backlog cost				
	1	2	0	1	2	Total cost	Δ (%)
SP	2.368	2.277	0.634	1.961	0.352	7.592	24.0
LSP	5.989	2.921	0.193	0.865	0.246	10.213	66.9

base stock levels, 41 for component 1 and 30 for component 2. Table 2 shows the inventory costs of using these two different sets of base stock levels. By allowing some modest increase of backlog costs, SP achieves large savings in the inventory holding costs. Its total inventory cost is only 3/4 of that under LSP and its optimality gap (the lower bound is 6.12) is slightly above 1/3 of that of LSP.

The remaining 56 test cases are in region A, and SP outperforms LSP in 34 of them (see Table 3). We have also compared SP with the aforementioned SPP policy (the same as SP except that no component is held back for product 0), and similarly find SP is better in 33 cases. These results suggest that reserving components for a more valuable product not only is asymptotically optimal but also improves performance in a non-asymptotic regime. Nevertheless, the advantage of reservation is not uniform across all of the arrival rate combinations we considered. SP dominates LSP in 25 out of 28 cases where the arrival rate of product 0 does not exceed those of products 1 and 2 ($\lambda_0 = 10, 20$, $\lambda_1 = \lambda_2 = 20$). However, LSP has a smaller optimality gap than SP in all 14 cases where $\lambda_0 = \lambda_1 = 20$ and $\lambda_2 = 10$, and the same is true with SPP.

When a component 1 is held back from product 1, the system incurs an additional inventory cost c_1. The gain from paying this price is that when a component 2 becomes available, the backlog of product 0 can be cleared immediately, resulting in a larger reduction of the inventory cost. Under a base stock policy, replenishments of component 2 are triggered by arrivals of its demand, so a low arrival rate of product 2 demand means fewer replenishments of component 2 over time. As a result, a reserved component 1 needs to wait for a longer period of time before it can be used to serve product 0, making reservation more costly and thus damaging the performance of SP. The similar effect applies to cases where $\lambda_0 = 20$, $\lambda_1 = 10$ and $\lambda_2 = 20$. However, in the latter cases, it is the reserved component 2 that needs to wait for a longer period of time and the waiting costs c_2 ($<c_1$). Consequently, as is shown in the corresponding column of Table 3, SP remains a better policy than LSP in a majority (9 out of 14) of cases.

More to this point, holding back components sacrifices myopic optimality for future cost reductions. The immediate loss is more significant when the supply of reserved component is relatively more scarce, which under a base stock policy, happens under some asymmetric demand arrivals. Whether the overall outcome is better under reservation depends on the influence of current component availability on future states, the strength of which increases with the lead time. We illustrate this point in Table 4, by comparing SP with SPP, which are the same polices except that SPP does not reserve components. All cases feature asymmetric demand arrival rates with $\lambda_0 = \lambda_1 = 20$ and $\lambda_2 = 10$. With a short lead time $L = 1$, sacrificing optimality of the current component allocation yields little saving of future costs. Thus SPP outperforms SP in all cases. However, the conclusion is completely reversed when we increase L to 10.

6. Conclusions

Finding an optimal control policy for ATO inventory systems is both important and difficult. In this study,

Table 3 Comparisons in Region A: SP vs. LSP

					$(\lambda_0, \lambda_1, \lambda_2)$			
h_1	h_2	b_0	b_1	b_2	(20, 20, 20) $\Delta_{SP} - \Delta_{LSP}$ (%)	(10, 20, 20) $\Delta_{SP} - \Delta_{LSP}$ (%)	(20, 10, 20) $\Delta_{SP} - \Delta_{LSP}$ (%)	(20, 20, 10) $\Delta_{SP} - \Delta_{LSP}$ (%)
1	1	5.85	2.9	1.6	−2.47	0.24	−3.06	1.68
1	1	8.775	4.25	2.5	0.06	−1.95	0.16	1.53
0.5	1	3.25	1.9	0.6	−3.31	−3.36	−3.52	4.19
0.5	1	6.5	3.4	1.6	−1.72	−1.63	−1.70	1.63
0.5	1	9.425	4.75	2.5	−0.60	−1.04	1.68	4.15
1.5	1	5.2	2.4	1.6	−1.55	−0.15	0.36	0.41
1.5	1	8.125	3.75	2.5	−0.55	−1.80	−1.07	1.68
1	1	7.54	4.2	1.6	−0.51	−0.67	−1.01	3.16
1	1	11.05	6	2.5	−0.07	−0.48	−0.19	5.05
0.5	1	4.29	2.7	0.6	−3.57	−5.04	−3.73	5.78
0.5	1	8.19	4.7	1.6	−2.23	−0.26	−2.68	2.94
0.5	1	11.7	6.5	2.5	1.80	−2.74	1.52	5.85
1.5	1	6.89	3.7	1.6	−1.10	−1.81	−1.67	1.79
1.5	1	10.4	5.5	2.5	−0.71	−0.74	1.34	2.95

Table 4 Comparisons in Region A: SP vs. SPP

					$\lambda_0 = 20,\ \lambda_1 = 20,\ \lambda_2 = 10$			
					$L = 1$		$L = 10$	
h_1	h_2	b_0	b_1	b_2	Δ_{SP} (%)	Δ_{SPP} (%)	Δ_{SP} (%)	Δ_{SPP} (%)
1	1	5.85	2.9	1.6	15.9	14.5	7.7	8.6
1	1	8.775	4.25	2.5	15.4	14.3	6.9	7.6
0.5	1	3.25	1.9	0.6	34.2	31.2	13.7	14.5
0.5	1	6.5	3.4	1.6	20.9	19.6	10.2	10.7
0.5	1	9.425	4.75	2.5	20.1	19.0	8.9	9.3
1.5	1	5.2	2.4	1.6	14.1	12.6	6.8	7.8
1.5	1	8.125	3.75	2.5	13.7	12.6	6.1	6.9
1	1	7.54	4.2	1.6	19.0	17.3	9.3	10.0
1	1	11.05	6	2.5	18.6	17.3	8.2	8.8
0.5	1	4.29	2.7	0.6	37.3	34.4	17.4	18.0
0.5	1	8.19	4.7	1.6	26.3	24.7	12.8	13.2
0.5	1	11.7	6.5	2.5	25.2	24.0	11.1	11.4
1.5	1	6.89	3.7	1.6	15.9	14.2	7.8	8.6
1.5	1	10.4	5.5	2.5	15.5	14.1	6.9	7.5

we continue the exploration of an SP-based approach that was introduced in Doğru et al. (2010) and expanded upon in Reiman and Wang (2015). Like these two previous works, we focus on systems with identical lead times. We study structural properties of corresponding SPs, the conventional one-period ATO model and its relaxation, and make performance comparisons that use the M system as the testbed. Our contributions and their implications for future work are summarized as follows.

1. Zipkin (2016) shows that one-period ATO models associated with a polymatroid structure can be tackled effectively because they are cover-$L^\natural$-convex. But his analysis leaves out many other cases. By defining the chained-BOM structure and proving that models in that family are $L^\natural$ convex, we expand the reach of efficient algorithms to a new class of systems. Since $L^\natural$ convexity is a stronger property than cover-$L^\natural$-convexity, problems in our chained BOM family can be solved more conveniently by the direct use of the steepest-decent algorithm to search for the global optimum.

Of course, many ATO systems are neither in the tree family nor have a chained BOM, especially those that involve non-binary usage of components. As we show in section 3.3, these systems commonly embed a Knapsack model, so it is hopeless to search for structural properties that admit efficient solution procedures. How to take alternative paths to address these systems is an interesting challenge.

2. Zipkin (2016) bases his analysis on the fact that the second-stage problem of the SP has an explicit and greedy optimal solution when it is associated with a polymatroid constraint set. The M system does not have this property and yet we are able to derive an explicit form of the optimal solutions. This development enables us to reduce the two-stage SPs to one-stage optimization problems, making the entire SPs much easier to solve. More importantly, it also simplifies the SP-based allocation policy into a set of simple rules, which paves the way for implementation.

As is exemplified by product 0 (in region A) and product 1 (in regions C and D) in the M system, in the absence of the polymatroid structure, the optimal solution of the second-stage problem is partially greedy when some high-value products consume less resources. In these cases, we can immediately fix the greedy part of the optimal solution and derive the remaining values by solving a lower-dimension problem. Our derivation of the explicit solution for the M system benefits from this simple and intuitive step. Moreover, greedy solutions also imply static priority or reservation policies for component allocation, which is easy to implement. These findings can be adopted to simplify the optimization of SPs and facilitate the implementation of the allocation policy for larger and more complicated ATO systems.

3. The study features an extensive numerical study with the M system as the testbed. It shows that the optimality gap of the SP-based policy converges to 0 as the lead time grows while other prevailing approaches in the literature, built upon FIFO or No Holdback allocations, appear to fail to do so. The observation is consistent with proven results on asymptotic optimality in general or analogous systems in the past work (Reiman and Wang 2015, Wan and Wang 2015). Outside the asymptotic regime, the SP-based approach also outperforms these alternative policies in a majority, if not all, test cases.

Granted, the M system is a stylized model that does not involve as many parts and products that one may encounter in practice. Nevertheless, our numerical results do have implications for more general systems. For instance,

(a) For the related ATO production-inventory systems, component reservation (or rationing) is necessary for exactly optimal allocation in some cases (see Benjaafar and ElHafsi 2006, Nadar et al. 2014) or asymptotically optimal allocation in some other cases (Wan and Wang 2015). While component replenishment of ATO inventory systems differs from the above systems (ordered with delays instead of produced with capacity constraints), the allocation problem is similar. Analogous reasoning suggests that reservation is also necessary and our numerical results confirm this conclusion. Moreover, following the discussion in section 5.1, the simple structure of the M system crystallizes situations when the need for reservation arises: a highly valuable product uses multiple components consumed individually by many low-value items (region A). Comparable cases are ubiquitous in more complicated systems. The need for reservation may be clouded by other confounding factors there, but the outcome from the M system indicates it should stay in the playbook of inventory management.

(b) The development of the SP-based approach is not yet finished. In particular, how to manage ATO systems with non-identical lead times remains a challenge. It is well known that base stock policies are generally inadequate (Zipkin 2000). Extending their work on identical lead time systems, Reiman and Wang (2012) generalize the two-stage SPs here to a multi-stage SP and describe how to use their solutions to formulate a non-base stock replenishment policy. They prove that the policy coincides with the optimal policy of Rosling (1989) for single-product systems and is optimal for a special multi-product system where the allocation decision is trivial. Nevertheless, to complete this effort for general systems requires further policy specification, development and refinement, and proofs of optimality properties. The success with the original approach for systems with identical lead times, shown by its commanding lead over alternative policies in our numerical testing of the M system, gives assurance that exploring this path is a worthy effort.

4. As mentioned in the Introduction, the M system is commonly used as a testbed for different approaches. For future researchers who want to develop a better procedure and evaluate it on the same testbed, our SP-based policy facilitates their task by being a suitable benchmark. While surpassing this benchmark is likely to be challenging, as is suggested by comparisons in section 5, making comparisons should be easy.

To elaborate on the latter point, consider the well-characterized base stock replenishment and FIFO allocation policy in Lu and Song (2005), which can also be used as a candidate for comparison. In fact Lu and Song (2005) does use the M system for comparison, and observe that their objective function takes a complex form that makes exact calculation "rather difficult." Our experience with numerical comparisons in section 5 confirms this point. In contrast, because of developments in Sections 3 and 4, specifying our SP-based policy for the M system takes little effort that only involves solving a one-stage discrete optimization problem with desirable properties that allow optimal base stock levels to be determined efficiently by a common search procedure, and following some commonly understood rules to allocate components.

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Appendix. Proofs and Algorithms

A.1. Proof of Lemma 1

Let (z_0^*, z_1^*, z_2^*) be the optimal solution of the original SP. Then

$$\begin{aligned} z_i^* &= D_i \wedge (y_i - z_0^*) = D_i - [D_i - y_i + z_0^*]^+ \\ &= D_i - [z_0^* - \tilde{y}_i]^+, i = 1, 2, \end{aligned} \tag{A1}$$

where $0 \le z_0^* \le y_1 \wedge y_2 \wedge D_0$.

Setting z_i^* (i = 1, 2) higher is not feasible and setting it lower is not optimal. Applying the above to the original SP reduces the recourse LP to

$$\max_{z_0 \ge 0} \{\Gamma(z_0) + c_1 D_1 + c_2 D_2 | z_0 \le D_0 \wedge y_1 \wedge y_2\}$$

where $\Gamma(z_0) = c_0 z_0 - c_1(z_0 - \tilde{y}_1^+)^+ - c_2(z_0 - \tilde{y}_2^+)^+$,

- If $c_1 + c_2 \le c_0$, then $\Gamma(z_0)$ always increases in z_0, so $z_0^* = D_0 \wedge y_1 \wedge y_2$.
- If $c_1 \le c_0 < c_1 + c_2$, then $\Gamma(z_0)$ increases in z_0 if and only if $z_0 \le \tilde{y}_1^+$ or $z_0 \le \tilde{y}_2^+$, so $z_0^* = D_0 \wedge [(\tilde{y}_1^+ \vee \tilde{y}_2^+) \wedge y_1 \wedge y_2]$.
- If $c_2 \le c_0 < c_1$, then $\Gamma(z_0)$ increases in z_0 if and only if $z_0 \le \tilde{y}_1^+$, so $z_0^* = D_0 \wedge \tilde{y}_1^+ \wedge y_2$.
- If $c_0 < c_2$, then $\Gamma(z_0)$ increases in z_0 if and only if $z_0 \le \tilde{y}_1^+ \wedge \tilde{y}_2^+$, so $z_0^* = D_0 \wedge \tilde{y}_1^+ \wedge \tilde{y}_2^+$.

In the relaxed SP, (A1) also applies except that z_i (i = 0, 1, 2) can be negative, so $z_0 \ge 0$ and $z_0 \le y_1 \wedge y_2$ can be removed. The recourse LP reduces to

$$\max_{z_0} \{c_0 z_0 - c_1(z_0 - \tilde{y}_1)^+ - c_2(z_0 - \tilde{y}_2)^+ | z_0 \le D_0\},$$

and values of z_0^* in Table 1 are obtained from the same reasoning as above. □

A.2. Solving the SPs for the M System

By developing explicit formulas to evaluate $E[z_i^*(\mathbf{y}; \mathbf{D})]$ (i = 0, 1, 2), we can avoid sampling-based approximations and optimize both the original and relaxed SPs *exactly* for the M system. Lemma 1 gives $z_i^*(\mathbf{y}; \mathbf{D})$ (i = 0, 1, 2) on each sample path, but taking expected values of these pathwise solutions is still a substantial (and sometimes tedious) task. Below we present our calculation.

We use as inputs the density and cumulative probability functions of demands, $f_i(k)$, $F_i(k)$ (i = 0, 1, 2, k = 0, 1, ...). To simplify expressions, we extend the domain of the cumulative probability functions and define

$$F_i(k) = 0 \text{ and } \bar{F}_i(k) = 1 \text{ for } k = -1, -2, \ldots (i = 0, 1, 2).$$

We will also treat truncated demand expectations $E[D_i \wedge y]$ (where y is a given (integer) constant, i = 0, 1, 2) as given. For Poisson or Compound Poisson distributions, these values are easy to calculate and can be pre-computed before the optimization starts.

Recall that $\tilde{y}_i$ denotes $y_i - D_i$ (i = 1, 2). It is easy to check that for any x,

$$D_i \wedge (y_i - \tilde{y}_i \wedge x) = D_i (i = 1, 2) \quad \text{(A2)}$$

$$D_i \wedge (y_i - \tilde{y}_i^+ \wedge x) = D_i \wedge y_i (i = 1, 2). \quad \text{(A3)}$$

We will use both equations in our discussions.

• Region A: $c_1 + c_2 < c_0$,

For the original SP, Lemma 1 shows that on each sample path,

$$\begin{aligned} z_0^* &= D_0 \wedge y_1 \wedge y_2, \\ z_1^* &= D_1 \wedge (y_1 - D_0 \wedge y_1 \wedge y_2) \\ &= \begin{cases} D_1 \wedge (y_1 - D_0) & \text{if } D_0 < y_1 \wedge y_2 \\ D_1 \wedge (y_1 - y_2)^+ & \text{if } D_0 \geq y_2 \wedge y_1, \end{cases} \\ z_2^* &= D_2 \wedge (y_2 - D_0 \wedge y_1 \wedge y_2) \\ &= \begin{cases} D_2 \wedge (y_2 - D_0) & \text{if } D_0 < y_1 \wedge y_2 \\ D_2 \wedge (y_2 - y_1)^+ & \text{if } D_0 \geq y_1 \wedge y_2. \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} E[z_0^*] &= E[D_0 \wedge y_1 \wedge y_2], \\ E[z_1^*] &= \sum_{k_0=0}^{y_1 \wedge y_2} f_0(k_0) E[D_1 \wedge (y_1 - k_0)] + \bar{F}_0(y_1 \wedge y_1) \\ &\quad \times E[D_1 \wedge (y_1 - y_2)^+], \\ E[z_2^*] &= \sum_{k_0=0}^{y_1 \wedge y_2} f_0(k_0) E[D_2 \wedge (y_2 - k_0)] + \bar{F}_0(y_1 \wedge y_2) \\ &\quad \times E[D_2 \wedge (y_2 - y_1)^+]. \end{aligned}$$

For the relaxed SP, the lemma shows that $z_0^* = D_0$ and $z_i^* = D_i \wedge (y_i - D_0) (i = 1, 2)$, so $E[z_0^*] = E[D_0]$ and $E[z_i^*] = \sum_{k_0=0}^{\infty} f_0(k_0) E[D_i \wedge (y_i - k_0)] (i = 1, 2)$.

• Region B ($c_2 \leq c_1 < c_0 \leq c_1 + c_2$)

This is the most complicated case. To shorten our discussion, we will only consider the case where $y_1 \geq y_2$. Products 1 and 2 are symmetric in this case, so we can apply the same formula with indices 1 and 2 switched when $y_1 < y_2$.

By Lemma 1, in the original SP (observe that $y_1 \geq y_2$ implies $y_1 \geq \tilde{y}_2$),

$$\begin{aligned} z_0^* &= D_0 \wedge [(\tilde{y}_1^+ \wedge y_2) \vee (y_1 \wedge \tilde{y}_2^+)] \\ &= D_0 \wedge [(\tilde{y}_1^+ \vee \tilde{y}_2^+) \wedge y_2] \end{aligned}$$

which gives rise to the following three scenarios.

1. If $\tilde{y}_1^+ < \tilde{y}_2^+ \leq y_2$, then $y_2 > D_2$, $(\tilde{y}_1^+ \vee \tilde{y}_2^+) \wedge y_2 = \tilde{y}_2^+$. If $\tilde{y}_1^+ > D_0$, then

$$D_1 \wedge (y_1 - D_0 \wedge \tilde{y}_2^+) > D_1 \wedge (y_1 - \tilde{y}_1^+ \wedge \tilde{y}_2^+) = D_1,$$

where the last equality comes from (A3). If $\tilde{y}_1^+ \leq D_0 \wedge \tilde{y}_2^+$, then

$$D_1 \geq D_1 + \tilde{y}_1^+ - D_0 \wedge \tilde{y}_2^+ \geq y_1 - D_0 \wedge \tilde{y}_2^+.$$

Therefore,

$$\begin{aligned} z_0^* &= D_0 \wedge \tilde{y}_2^+, \\ z_1^* &= D_1 \wedge (y_1 - D_0 \wedge \tilde{y}_2^+) \\ &= \begin{cases} D_1 & \text{if } D_0 < \tilde{y}_1^+ \\ y_1 - D_0 & \text{if } \tilde{y}_1^+ \leq D_0 \leq \tilde{y}_2^+ \\ y_1 - \tilde{y}_2^+ & \text{if } \tilde{y}_2^+ < D_0, \end{cases} \\ z_2^* &= D_2 \wedge (y_2 - D_0 \wedge \tilde{y}_2^+) = D_2 \wedge y_2 = D_2. \end{aligned}$$

2. If $\tilde{y}_2^+ \leq \tilde{y}_1^+ < y_2$, then $(\tilde{y}_1^+ \vee \tilde{y}_2^+) \wedge y_2 = \tilde{y}_1^+ \geq 0$. Similar to the above,

$$\begin{aligned} z_0^* &= D_0 \wedge \tilde{y}_1^+, \\ z_1^* &= D_1 \wedge (y_1 - D_0 \wedge \tilde{y}_1^+) \\ &= D_1 \wedge y_1, \\ z_2^* &= D_2 \wedge (y_2 - D_0 \wedge \tilde{y}_1^+) \\ &= \begin{cases} D_2 & \text{if } D_0 < \tilde{y}_2^+ \\ y_2 - D_0 & \text{if } \tilde{y}_2^+ \leq D_0 \leq \tilde{y}_1^+ \\ y_2 - \tilde{y}_1^+ & \text{if } \tilde{y}_1^+ < D_0. \end{cases} \end{aligned}$$

3. Finally, if $y_2 \leq \tilde{y}_1^+$, then $0 \leq D_1 \leq y_1 - y_2$, $(\tilde{y}_1^+ \vee \tilde{y}_2^+) \wedge y_2 = y_2$, and

$$\begin{aligned} &z_0^* = D_0 \wedge y_2, z_1^* = D_1 \wedge (y_1 - D_0 \wedge y_2) = D_1, \\ &z_2^* = D_2 \wedge (y_2 - D_0 \wedge y_2) = D_2 \wedge (y_2 - D_0)^+. \end{aligned}$$

Summarizing these three situations,

$$\begin{aligned} E[z_0^*] = &\sum_{k_2=0}^{y_2} f_2(k_2) \bar{F}_1(y_1 - y_2 + k_2) E[D_0 \wedge (y_2 - k_2)] \\ &+ \sum_{k_1=y_1-y_2+1}^{y_1} f_1(k_1) \bar{F}_2(y_2 - y_1 + k_1 - 1) \\ &E[D_0 \wedge (y_1 - k_1)] + F_1(y_1 - y_2) E[D_0 \wedge y_2], \end{aligned}$$

where three terms correspond to the above three situations. The value of $E[z_1^*]$ is given by a lengthy formula, which we will explain after presenting the equation.

$$E[z_1^*] =$$

$$\bar{F}_1(y_1) \left[\sum_{k_2=0}^{y_2} f_2(k_2) (y_1 - E[D_0 \wedge (y_2 - k_2)]) + y_1 \bar{F}_2(y_2) \right]$$

$$+\sum_{k_1=y_1-y_2+1}^{y_1} k_1 f_1(k_1)F_2(y_2-y_1+k_1-1)F_0(y_1-k_1-1)$$
$$+\sum_{k_1=y_1-y_2+1}^{y_1} f_1(k_1)\sum_{k_2=0}^{y_2-y_1+k_1-1} f_2(k_2)\sum_{k_0=y_1-k_1}^{y_2-k_2}(y_1-k_0)f_0(k_0)$$
$$+\sum_{k_1=y_1-y_2+1}^{y_1} f_1(k_1)\sum_{k_2=0}^{y_2-y_1+k_1-1}(y_1-y_2+k_2)f_2(k_2)\bar{F}_0(y_2-k_2)$$
$$+\sum_{k_1=y_1-y_2+1}^{y_1} k_1 f_1(k_1)\bar{F}_2(y_2-y_1+k_1)+\sum_{k_1=1}^{y_1-y_2+1} k_1 f_1(k_1)$$

The first line on the right-hand side captures the case when $y_1 < D_1$, which can only happen in situation 1 when $\tilde{y}_1^+ \le D_0$ and in situation 2 when $y_2 \le D_2$. The two terms in the square bracket represents the case when $y_2 \ge D_2$ and $y_2 < D_2$ respectively. For cases when $y_1 \ge D_1$, the second, third, and fourth lines correspond to three cases in situation 1, and the last two terms correspond to situations 2 and 3 respectively.

In the following evaluation of $E[z_2^*]$, $y_1 < D_1$ can happen only in situation 1, resulting in $z_2^* = D_2 \wedge y_2$, or in situation 2 when $\tilde{y}_1^+ \le D_0$ (the equality holds when $D_0 = 0$) and $D_2 \ge y_2$, which also leads to $z_2^* = y_2 = D_2 \wedge y_2$. For cases when $y_1 \ge D_1$, the second term on the right corresponds to situation 1, the third, fourth, and fifth terms correspond to three cases in situation 2, and the last term corresponds to situation 3.

$$E[z_2^*] = \bar{F}_1(y_1)E[D_2 \wedge y_2]$$
$$+\sum_{k_1=y_1-y_2+1}^{y_1} f_1(k_1)\sum_{k_2=0}^{y_2-y_1+k_1-1} k_2 f_2(k_2)$$
$$+\sum_{k_1=y_1-y_2+1}^{y_1} f_1(k_1)\sum_{k_2=y_2-y_1+k_1}^{y_2} k_2 f_2(k_2)F_0(y_2-k_2)$$
$$+\sum_{k_1=y_1-y_2+1}^{y_1} f_1(k_1)\sum_{k_0=0}^{y_1-k_1}(y_2-k_0)f_0(k_0)\bar{F}_2(y_2-k_0)$$
$$+\sum_{k_1=y_1-y_2+1}^{y_1}(y_2-y_1+k_1)f_1(k_1)\bar{F}_0(y_1-k_1)\bar{F}_2(y_2-y_1+k_1-1)$$
$$+F_1(y_1-y_2)\sum_{k_0=0}^{y_2} f_0(k_0)E[D_2 \wedge (y_2-k_0)]$$

Evaluating $E[z_i^*]$ (i = 0, 1, 2) for the relaxed SP is a little simpler.

1. If $\tilde{y}_2 \le \tilde{y}_1$, then

$$z_0^* = D_0 \wedge (\tilde{y}_1 \vee \tilde{y}_2) = D_0 \wedge \tilde{y}_1,$$
$$z_1^* = D_1 \wedge (y_1 - D_0 \wedge \tilde{y}_1) = D_1 \ (using(A2)),$$
$$z_2^* = D_2 \wedge (y_2 - D_0 \wedge \tilde{y}_1) = \begin{cases} D_2 & \text{if } D_0 < \tilde{y}_2 \le \tilde{y}_1 \\ y_2 - D_0 & \text{if } \tilde{y}_2 \le D_0 \le \tilde{y}_1 \\ y_2 - \tilde{y}_1 & \text{if } \tilde{y}_2 \le \tilde{y}_1 < D_0 \end{cases}$$

- If $\tilde{y}_1 < \tilde{y}_2$, then

$$z_0^* = D_0 \wedge (\tilde{y}_1 \vee \tilde{y}_2) = D_0 \wedge \tilde{y}_2,$$
$$z_1^* = D_1 \wedge (y_1 - D_0 \wedge \tilde{y}_2) = \begin{cases} D_1 & \text{if } D_0 < \tilde{y}_1 < \tilde{y}_2 \\ y_1 - D_0 & \text{if } \tilde{y}_1 < D_0 \le \tilde{y}_2 \\ y_1 - \tilde{y}_2 & \text{if } \tilde{y}_1 < \tilde{y}_2 < D_0 \end{cases},$$
$$z_2^* = D_2 \wedge (y_2 - D_0 \wedge \tilde{y}_2) = D_2 \ (using(A2)).$$

By summarizing the above two cases,

$$E[z_0^*] = \sum_{k_1=0}^{\infty} f_1(k_1)\bar{F}_2(y_2-y_1+k_1-1)E[D_0 \wedge (y_1-k_1)]$$
$$+\sum_{k_2=0}^{\infty} f_2(k_2)\bar{F}_1(y_1-y_2+k_2)E[D_0 \wedge (y_2-k_2)].$$

When evaluating $E[z_1^*]$, we observe that $D_0 \le \tilde{y}_i$ only if $D_i \le y_i$ (i = 1, 2), which explains why in the following, some summations over k_i (i = 1, 2) are carried to $+\infty$ while others are up to y_i (i = 1, 2).

$$E[z_1^*] = \sum_{k_1=0}^{+\infty} k_1 f_1(k_1)\bar{F}_2(y_2-y_1+k_1-1)$$
$$+\sum_{k_2=0}^{y_2} f_2(k_2)\sum_{k_1=y_1-y_2+k_2+1}^{y_1} k_1 f_1(k_1)F_0(y_1-k_1)$$
$$+\sum_{k_2=0}^{y_2} f_2(k_2)\sum_{k_1=y_1-y_2+k_2+1}^{+\infty} f_1(k_1)$$
$$\sum_{k_0=y_1-k_1+1}^{y_2-k_2}(y_1-k_0)f_0(k_0)+\sum_{k_2=0}^{+\infty} f_2(k_2)$$
$$(y_1-y_2+k_2)\bar{F}_1(y_1-y_2+k_2)\bar{F}_0(y_2-k_2)$$

In the following evaluation of $E[z_2^*]$, the assumption that $y_1 \ge y_2$ makes it necessary to use $(y_2 - y_1 + 1)^+$ instead of $y_2 - y_1 + 1$ as the starting point of some summations.

$$E[z_2^*] = \sum_{k_1=0}^{y_1} f_1(k_1)\sum_{k_2=(y_2-y_1+k_1)^+}^{y_2} k_2 f_2(k_2)F_0(y_2-k_2)$$
$$+\sum_{k_1=0}^{y_1} f_1(k_1)\sum_{k_2=(y_2-y_1+k_1)^+}^{+\infty} f_2(k_2)$$
$$\sum_{k_0=y_2-k_2+1}^{y_1-k_1}(y_2-k_0)f_0(k_0)+\sum_{k_1=0}^{+\infty}(y_2-y_1+k_1)f_1(k_1)$$
$$\bar{F}_2(y_2-y_1+k_1-1)\bar{F}_0(y_1-k_1)$$
$$+\sum_{k_2=0}^{+\infty} k_2 f_2(k_2)\bar{F}_1(y_1-y_2+k_2)$$

• Region C ($c_2 < c_0 \leq c_1$)
In the original SP,

$$z_0^* = D_0 \wedge \tilde{y}_1^+ \wedge y_2 = \begin{cases} D_0 \wedge y_2 & \text{if } y_2 < y_1 - D_1 \\ D_0 \wedge (y_1 - D_1)^+ & \text{if } y_2 \geq y_1 - D_1 \end{cases}$$

$$z_1^* = D_1 \wedge (y_1 - z_0^*) = D_1 \wedge y_1,$$

$$z_2^* = D_2 \wedge (y_2 - z_0^*) = \begin{cases} 0 & \text{if } \tilde{y}_1^+ = 0 \text{ or } D_0 \geq \tilde{y}_1^+ > y_2 \\ D_2 \wedge (y_2 - D_0) & \text{if } D_0 \leq \tilde{y}_1^+ \wedge y_2 \\ D_2 \wedge (y_2 - \tilde{y}_1^+)^+ & \text{if } D_0 > y_2 \geq \tilde{y}_1^+. \end{cases}$$

Therefore,

$$E[z_0^*] = F_1(y_1 - y_2 - 1)E[D_0 \wedge y_2] + \sum_{k_1=(y_1-y_2)^+}^{y_1} f_1(k_1)E[D_0 \wedge (y_1 - k_1)],$$

$$E[z_1^*] = E[D_1 \wedge y_1].$$

To find $E[z_2^*]$, we divide $D_0 \leq y_2 \wedge \tilde{y}_1^+$ into regions where $D_0 \leq y_2 < y_1 - D_1$ and $D_0 < y_1 - D_1 \leq y_2$,

$$E[z_2^*] = F_1(y_1 - y_2 - 1)\sum_{k_0=0}^{y_2} f_0(k_0)E[D_2 \wedge (y_2 - k_0)] + \sum_{k_1=(y_1-y_2)^+}^{y_1} f_1(k_1) \sum_{k_0=0}^{y_1-k_1-1} f_0(k_0)E[D_2 \wedge (y_2 - k_0)] + \bar{F}_0(y_2) \sum_{k_1=(y_1-y_2+1)^+}^{+\infty} f_1(k_1)E[D_2 \wedge (y_2 - (y_1 - k_1)^+)].$$

In the relaxed SP,

$$z_0^* = D_0 \wedge \tilde{y}_1, \; z_1^* = D_1 \wedge (y_1 - D_0 \wedge \tilde{y}_1) = D_1 \text{ (using (A2))},$$

$$z_2^* = D_2 \wedge (y_2 - D_0 \wedge \tilde{y}_1) = \begin{cases} D_2 \wedge (y_2 - \tilde{y}_1) & \text{if } D_0 > \tilde{y}_1 \\ D_2 \wedge (y_2 - D_0) & \text{if } D_0 \leq \tilde{y}_1 \end{cases}.$$

Notice in the formula of z_2^*, $D_0 \leq \tilde{y}_1$ implies $D_1 \leq y_1$, so

$$E[z_0^*] = \sum_{k_1=0}^{+\infty} f_1(k_1)E[D_0 \wedge (y_1 - k_1)], \; E[z_1^*] = E[D_1],$$

$$E[z_2^*] = \sum_{k_1=0}^{+\infty} f_1(k_1)\bar{F}_0(y_1 - k_1)E[D_2 \wedge (y_2 - y_1 + k_1)] + \sum_{k_1=0}^{y_1} f_1(k_1) \sum_{k_0=0}^{y_1-k_1} f_0(k_0)E[D_2 \wedge (y_2 - k_0)].$$

• Region D ($c_0 \leq c_2 \leq c_1$)

$$z_0^* = D_0 \wedge \tilde{y}_1^+ \wedge \tilde{y}_2^+ = \begin{cases} D_0 \wedge \tilde{y}_1^+ & \text{if } \tilde{y}_1^+ \leq \tilde{y}_2^+ \\ D_0 \wedge \tilde{y}_2^+ & \text{if } \tilde{y}_1^+ \geq \tilde{y}_2^+ \end{cases}$$

$$z_1^* = D_1 \wedge (y_1 - D_0 \wedge \tilde{y}_1^+ \wedge \tilde{y}_2^+) = D_1 \wedge y_1,$$

$$z_2^* = D_2 \wedge (y_2 - D_0 \wedge \tilde{y}_1^+ \wedge \tilde{y}_2^+) = D_2 \wedge y_2,$$

so $E[z_1^*] = E[D_1 \wedge y_1]$, $E[z_2^*] = E[D_2 \wedge y_2]$.

To determine $E[z_0^*]$, notice that $\tilde{y}_1^+ \leq \tilde{y}_2^+$ applies only to cases where $D_1 \geq y_1 - y_2$ when $y_1 \geq y_2$, and may hold for cases where $D_1 \geq 0$ when $y_1 < y_2$. Similarly, for $\tilde{y}_2^+ < \tilde{y}_1^+$ to hold, it is necessary that $D_2 > y_2 - y_1$ when $y_2 > y_1$, $D_2 > 0$ when $y_2 = y_1$, and $D_2 \geq 0$ when $y_2 < y_1$. Therefore,

$$E[z_0^+] = \sum_{k_1=(y_1-y_2)^+}^{y_1-1} f_1(k_1)F_2(y_2 - y_1 + k_1)E[(y_1 - k_1) \wedge D_0] + \sum_{k_2=(y_2-y_1+1)^+}^{y_2-1} F_1(y_1 - y_2 + k_2 - 1)f_2(k_2) \mathbf{E}[(y_2 - k_2) \wedge D_0].$$

The pathwise solution for the relaxed SP is as follows:

$$z_0^* = D_0 \wedge \tilde{y}_1 \wedge \tilde{y}_2 = \begin{cases} D_0 \wedge \tilde{y}_1 & \text{if } \tilde{y}_1 \leq \tilde{y}_2 \\ D_0 \wedge \tilde{y}_2 & \text{if } \tilde{y}_1 > \tilde{y}_2 \end{cases},$$

$$z_1^* = D_1 \wedge (y_1 - D_0 \wedge \tilde{y}_1 \wedge \tilde{y}_2) = D_1,$$

$$z_2^* = D_2 \wedge (y_2 - D_0 \wedge \tilde{y}_1 \wedge \tilde{y}_2) = D_2.$$

so $E[z_1^*] = E[D_1]$, $E[z_2^*] = E[D_2]$.

Without non-negativity constraint, $\tilde{y}_1 \leq \tilde{y}_2$ may hold for $D_1 \geq (y_1 - y_2)^+$ and $\tilde{y}_1 > \tilde{y}_2$ may hold for $D_2 \geq (y_2 - y_1 + 1)^+$, in which case

$$E[z_0^*] = \sum_{k_1=(y_1-y_2)^+}^{\infty} f_1(k_1)F_2(y_2 - y_1 + k_1)E[(y_1 - k_1) \wedge D_0] + \sum_{k_2=(y_2-y_1+1)^+}^{\infty} F_1(y_1 - y_2 + k_2 - 1)f_2(k_2) E[(y_2 - k_2) \wedge D_0].$$

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