MATHEMATICS OF OPERATIONS RESEARCH
Vol. 33, No. 2, May 2008, pp. 257–282
ISSN 0364-765X | EISSN 1526-5471 | 08 | 3302 | 0257

DOI 10.1287/moor.1070.0288

An Asymptotically Optimal Policy for a Quantity-Based Network Revenue Management Problem

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We consider a canonical revenue management problem in a network setting where the goal is to find a customer admission policy to maximize the total expected revenue over a fixed finite horizon. There is a set of resources, each of which has a fixed capacity. There are several customer classes, each with an associated arrival process, price, and resource consumption vector. If a customer is accepted, it effectively removes the resources that it consumes from the system. The exact solution cannot be obtained for reasonable-sized problems due to the curse of dimensionality. Several (approximate) solution techniques have been proposed in the literature. One way to analytically compare policies is via an asymptotic analysis where both resource sizes and arrival rates grow large. Many of the proposed policies are asymptotically optimal on the fluid scale. However, as we demonstrate in this paper, these policies may fail to be optimal on the more sensitive diffusion scale even for quite simple problem instances. We develop a new policy that achieves diffusion-scale optimality. The policy starts with a probabilistic admission rule derived from the optimization of the fluid model, embeds a trigger function that tracks the difference between the actual and expected customer acceptance, and sets threshold values for the trigger function, the violation of which invokes the reoptimization of the admission rule. We show that re-solving the fluid model, which needs to be performed at most once, is required for extending the asymptotic optimality from the fluid scale to the diffusion scale. We demonstrate the implementation of the policy by numerical examples.

Key words: revenue management; asymptotic optimality; admission control; diffusion limit; fluid limit; reoptimization
MSC2000 subject classification: Primary: 90B15, 90B05, 60K30; secondary: 60F05, 60F17
OR/MS subject classification: Primary: inventory and production: revenue management; secondary: probability; diffusion: limit theorems
History: Received November 2, 2005; revised November 3, 2006.

1. Introduction. In this paper, we investigate a canonical revenue management problem and develop a new approach that exhibits better performance for "large" problems than previous approaches. The *problem* is defined as follows: There is a set of resources with fixed capacities to be used by customers who arrive randomly during a fixed finite time interval. Customers are divided into different classes based on their usage of resources and the (fixed) price they pay for the service. Depending on their classes, arriving customers are either immediately accepted for service or rejected—no waiting or backlog is allowed. If a customer is accepted, it removes the resources that it consumes from the system. Unused resources at the end of the time interval have no salvage value. The objective is to find an admission policy to maximize the total expected revenue.

A classical example of this problem is airline seat inventory control, where a resource corresponds to a flight leg and capacity corresponds to the number of seats on the flight. Customer classes are defined by (itinerary, fare) combinations. The airline tries to allocate seats to different customer classes in a way that maximizes the expected revenue. Other instances of this problem can be found in various service industries such as car rental, hotel room reservation, and bandwidth provisioning in telecommunication networks. The same problem also exists in supply chain management, e.g., when an inventory manager needs to allocate some nonreplenishable stocks of components to different orders.

If the exact numbers of customer arrivals of each class over the time interval were given in advance, then the optimal number of customers of each class to accept could be obtained by solving an integer program. (In the asymptotic regime that interests us, the relaxation of this integer program to a linear program (LP) introduces asymptotically negligible error, so we only consider this relaxed LP, not the exact integer program. It should be noted that the optimal objective of the LP is an upper bound on the optimal objective of the integer program.) Of course, in normal situations the actual numbers of arrivals are not known until the process is over, so we refer to the expected revenue obtained under the above procedure as the *hindsight optimum*. Specifically, for each sample path of arrivals, let the hindsight revenue be the revenue that the decision maker could have achieved with ex post optimization of the admission decisions. The hindsight optimum is the expected value of the hindsight revenue. In this paper, we study ex ante policies that are determined based on limited information (mean, standard deviation, or distribution) about the arrival processes. The hindsight optimum sets the upper bound for expected revenue that can be achieved and is used as the reference point for measuring the performance of a given policy.

We develop our analysis under the following asymptotic framework: We introduce a sequence of problems of increasing size by scaling up customer arrivals and capacities by a factor of k ($k = 1, 2, \ldots$). We define the optimality gap of a given policy as the difference between the hindsight optimum and the expected revenue generated by the policy. We develop policies under which when problem size (k) increases, the optimality gap, after being normalized by some parameter related to k, diminishes to zero. A policy has a better asymptotic performance if the optimality gap can converge when a smaller normalizing parameter is used.

Previous research has shown that many existing policies are asymptotically optimal on the fluid scale, i.e., if the optimality gap is normalized by the problem size k, the resulting quantity converges to zero as k increases. However, as we demonstrate for an example (and conjecture more generally), these policies are not optimal on the diffusion scale; i.e., if the optimality gap is normalized by $\sqrt{k}$ instead of k, then the resulting value will not converge to zero. Our key contribution is to propose a new policy that achieves asymptotic optimality on the diffusion scale.

Our policy is developed from the observation that achieving hindsight optimality would have been trivial if the service provider could renege, i.e., take away resources from customers who have already been admitted or accept customers who have been previously rejected, or both. (Some proposed schemes in the literature, such as overbooking and options, give the service provider this flexibility at a cost.) While such reversal is not allowed in our problem, the service provider can still benefit from rebalance, i.e., to offset the effect of previous admissions by adjusting rules for accepting future arrivals. Our policy starts with an admission rule derived from solving an LP and, at an appropriate time, adjusts the rule by re-solving the LP with updated state information.

The revenue implication of re-solving a math program for updating control parameters has received considerable attention in recent studies. Normally, one would expect that re-solving should be beneficial because it makes use of more observations of customer arrivals. However, Cooper [3] illustrates a counterexample in which re-solving actually reduces the expected revenue. His case is studied in more detail by Secomandi [11] who shows that re-solving guarantees a (weak) revenue improvement under a certain condition that he terms the "sequential improvement property." In this paper, we demonstrate the necessity of re-solving by showing that

(i) without re-solving, no commonly used booking limit policies, which include the policy in Cooper [3], can be asymptotically optimal on the diffusion scale, regardless of how well these limits are preset; and

(ii) our policy, which re-solves the same LP model as in Cooper [3], is optimal on the diffusion scale. The analysis makes it evident that re-solving is crucial for reaching this optimality objective.

A related subject of interest is to decide the number of times to re-solve the math program and, more generally, when to re-solve it. The issue is posed as a topic for future research in Secomandi [11]. On the one hand, the adjustment should be made early enough so that there remain enough capacities and future arrivals to rebalance. On the other hand, it is desirable to keep the number of corrections to a minimum. In this paper, we show that it is sufficient to re-solve the LP model only once to achieve the diffusion-scale optimality, but the timing is critical. We develop a trigger-and-threshold mechanism to decide the re-solving time.

The development of asymptotically optimal policies sheds interesting insights on improving revenue performance for problems of all sizes. However, small-size problems sometimes give rise to what might best be termed round-off issues that can limit the effectiveness of an asymptotically optimal policy. In these cases, one can still use the same insight but build a different implementation heuristic. We discuss this issue with respect to our policy in the paper.

Our policy applies to general situations with few information and computation requirements. Some existing policies are derived for Poisson arrivals, a restriction we remove in our case. In §3, we outline conditions imposed on arrival processes, which apply at least to all renewal processes with finite second moment of interarrival times. Our policy only needs the knowledge about the mean arrival rate of each customer class, which is assumed to be constant over the interval. The calculation requires nothing beyond linear programming.

The rest of the paper is organized as follows. We present the mathematical formulation of the problem and review relevant literature in §2. In §3, we formalize the concept of asymptotic optimality and prove, for an example, that booking limit policies are not asymptotically optimal on the diffusion scale. In §4, we develop our policy and analyze its asymptotic properties. We demonstrate the advantages of the new policy via numerical examples and discuss implementation heuristics in §5.

2. Model formulation and previous work.

2.1. Notation, model, and the hindsight optimum. There are J customer classes indexed by $j = 1, 2, \ldots, J$ and L resources indexed by $l = 1, 2, \ldots, L$. Let a_{lj} denote the amount of resource l required to serve a class j customer, let r_j denote the price a class j customer pays for the service, and let C_l denote the capacity of resource l.

Customers arrive during a fixed time interval which, without loss of generality, we take to be $[0, 1]$. For $j = 1, 2, \ldots, J$, denote the number of class j customer arrivals in the period $[0, t]$ by $\Lambda_j(t)$. We provide our assumptions on the arrival processes (sufficient to obtain our asymptotic results) in §3. A concrete example covered by our assumptions is where $\{\Lambda_j(t), t \geq 0\}$, $1 \leq j \leq J$, are independent renewal processes with finite second moments of interarrival times. Let the random variables x_j denote the total number of class j customers accepted in the time interval $[0, 1]$. The following constraints must be satisfied:

$$0 \leq x_j \leq \Lambda_j(1), \quad 1 \leq j \leq J \qquad \text{and} \qquad \sum_j a_{lj} x_j \leq C_l, \quad 1 \leq l \leq L;$$

i.e., the number of customers accepted cannot exceed the number of customers arriving (arrival constraints) and the usage of a resource cannot exceed its capacity (capacity constraints). The expected revenue is $E[R] = E[\sum_j r_j x_j]$.

The hindsight optimum is defined to be $E[\bar{R}] \equiv E[\sum_j r_j \bar{x}_j]$ where $\bar{x}_j$ $(j = 1, 2, \ldots, J)$ solves the LP

$$\max_{x_j} \left\{ \sum_j r_j x_j \;\middle|\; \sum_j a_{lj} x_j \leq C_l,\, 0 \leq x_j \leq \Lambda_j(1) \right\}. \tag{1}$$

This typically unrealizable outcome is the standard by which we judge the performance of our policy.

2.2. Previous work. This revenue management problem has been studied extensively in the literature. Various techniques, such as mathematical programming, Markov decision processes, and optimal control, have been used to develop different admission schemes. We do not attempt to cite and discuss every significant contribution in this area. Instead we refer readers to Talluri and van Ryzin [13] for a comprehensive literature review. Here we focus on a few studies that are the most pertinent to our own work. They include works on booking limit, nesting, and bid price control. Common to our approach, all three schemes rely on solving mathematical programming models to define admission policies.

We also want to point out two papers on a related revenue management problem where dynamic pricing, rather than admission control is used. The first (to our knowledge) paper to consider asymptotic optimality in a revenue management context was Gallego and van Ryzin [6], who consider a single resource problem. They show that, with an appropriately chosen price, a fixed price policy is asymptotically optimal on the fluid scale. This is extended to the network setting in Gallego and van Ryzin [7].

2.2.1. Booking limit. Under this approach, the service provider sets a fixed quota for each customer class and accepts customers first-come-first-serve (FCFS) up to these limits. Actually, if these limits could be set by (1), then one would achieve the hindsight optimum. While it is not feasible to solve (1) in advance because $\Lambda_j(1)$, $j = 1, 2, \ldots, J$, are not known, a simple heuristic is to use $\lambda_j = E[\Lambda_j(1)]$ as proxies and set booking limits by the following "distinct deterministic model" (Williamson [15]):

$$\max_{x_j} \left\{ \sum_j r_j x_j \;\middle|\; \sum_j a_{lj} x_j \leq C_l,\, 0 \leq x_j \leq \lambda_j \right\}. \tag{2}$$

Despite the simplicity of the model, it is shown in Cooper [3] that using the solution of the "fluid LP" (2) is asymptotically optimal on the fluid scale (see §3.1).

The arrival constraints in (2) impose unnecessary restrictions on customer acceptance that may reduce total revenue. For instance, if $C_l = \infty$, then, to maximize revenue, all customers should be accepted. However, applying the booking limit set by (2) results in rejection of any arrival in excess of the mean. This limitation is avoided in Wollmer [16] where the booking limit is set by the following "Expected Marginal Revenue" or "Probabilistic Distinct" model:

$$\max_{x_j \geq 0} \left\{ \sum_j r_j E[\min(\Lambda_j(1),\, x_j)] \;\middle|\; \sum_j a_{lj} x_j \leq C_l \right\}. \tag{3}$$

The new objective function, $E[\sum_j r_j \min(\Lambda_j(1),\, x_j)]$, is the expected revenue achievable in the absence of capacity constraints, which is a concave increasing function of the booking limits. When the capacity becomes sufficiently loose, the model allows the booking limits to be set to ∞ so that every customer is accepted. Notice that solving (3) needs more information than solving (2). In (2), only the mean arrivals are used as inputs, while in (3) entire distributions of arrivals are needed to formulate the objective function.

As another twist of the above approach, a new optimization model is developed in Li and Yao [8, 9] in which capacity constraints are replaced with the following set of fixed-point approximations:

$$a_{lj}x_j + \sum_{j'\neq j} a_{lj'} E[\min(\Lambda_{j'}(1),\, x_{j'})] \leq C_l, \quad \forall\, j,\, l\colon a_{lj} > 0. \tag{4}$$

Using numerical examples, the authors demonstrate the superior performance of the policy that sets booking limits based on the refined optimization model.

2.2.2. Nesting. It is easy to envision situations in which booking limit policies become inefficient. Suppose there is one class of customers that pays a higher price for the service and uses fewer resources than another class. Let there be a sample path on which there exists a time when the number of arrivals of the first class exceeds its booking limit while the second class still has unused quota. The service provider gives up revenue if she rejects the former class and accepts the latter, which is exactly what can happen under a fixed booking limit policy. Nesting is a suggested remedy. Under this approach, customer classes are ranked based on the price and resource usage, and high-ranking classes are allowed to use quotas of low-ranking classes.

The nesting approach is shown to be effective for *single-leg problems*, where there is only one resource, so that, ignoring integrality, customer classes can be completely ranked based on price per unit of resource used (r_j/a_j, $j = 1, 2, \ldots, J$). The advantage of nesting becomes less clear in a networked problem where there are multiple resources. In this case, customer classes can typically only be partially ordered. (How do you rank two classes of customers if one uses more of resource 1 but less of resource 2 than the other?) While it is possible to come up with an integrated measure that forces a complete ranking, for each of these measures, it is always possible to construct a counterexample to show its disadvantage. For example, a simulation study in Williamson [15] shows that, among many ranking criteria, the use of the dual variables of the arrival constraints ($x_j \leq \lambda_j$) in (2) produces the best result. However, it is shown in a numerical example (Li and Yao [9]) that this scheme generates less revenue than the fixed booking limit policy in which no nesting is used.

2.2.3. Bid price control. Bid price control is yet another approach that sets an admission policy based on dual variables in the LP (2). A bid price μ_l is defined as the Lagrange multiplier attached to a capacity constraint ($\sum_j a_{lj}x_j \leq C_l$). All customers of a class will be accepted if the price that the class pays, r_j, is higher than the aggregated bid price $\sum_l a_{lj}\mu_l$, and they will be rejected if the price is lower. However, how to treat customers whose price exactly equals the aggregated bid price (i.e., $r_j = \sum_l a_{lj}\mu_l$) is a tricky question. When the admission depends only on the bid price, these customers have to be accepted or rejected entirely, but the optimal strategy may require accepting a fraction of these customers. As shown in Talluri and van Ryzin [12], there exist examples where the bid price control is suboptimal because it cannot properly handle the admission of customers in these boundary classes. Nevertheless, they also identify interesting cases where the policy does work well, which is when the price a customer pays follows a continuous distribution instead of taking on a finite number of distinct values. Consequently, there is a probability of zero that a customer's price equals the aggregated bid price, so the admission decisions of these customers have no impact on the total expected revenue.

3. The asymptotic framework. We introduce a sequence of problems, indexed by $k = 1, 2, \ldots,$ that we obtain by scaling up both the capacities and arrival rates with k. Thus we let

$$C_l^k = kC_l \quad l = 1, 2, \ldots, L, \tag{5}$$

where $0 < C_l < \infty$. We assume that there is a probability space $(\Omega, \mathcal{F}, P)$ on which the sequence of arrival processes is defined. We make two assumptions on the arrival process: The first assumption states that a functional central limit theorem holds for a properly centered and scaled sequence of arrival processes. Given a sequence of arrival processes $\{\Lambda^k, k \geq 1\}$, where $\Lambda^k = (\Lambda_1^k, \ldots, \Lambda_J^k)$ and $\Lambda_j^k = (\Lambda_j^k(t),\, 0 \leq t \leq 1)$, we define

$$\beta_j^k(t) = \frac{\Lambda_j^k(t) - kt\lambda_j}{\sqrt{k}}, \quad 1 \leq j \leq J, \quad 0 \leq t \leq 1, \text{ and } k \geq 1. \tag{6}$$

Let $\beta_j^k = (\beta_j^k(t), 0 \leq t \leq 1)$, $1 \leq j \leq J$ and $k \geq 1$, and $\beta^k = (\beta_1^k, \ldots, \beta_J^k)$, $k \geq 1$. Let $\beta = (\beta_1, \ldots, \beta_J)$, where $\beta_j = (\beta_j(t), 0 \leq t \leq 1)$, $1 \leq j \leq J$, denote a J dimensional driftless Brownian motion with covariance matrix $(\sigma_{ij}^2, 1 \leq i, j \leq J)$. Let $D^J[0,1]$ denote the space of right continuous functions with left limits defined on the time interval $[0,1]$ and taking values in $\mathbb{R}^J$ (endowed with the standard Skorohod topology). We let $\xrightarrow{d}$ denote convergence in distribution.

Assumption 3.1. *$\beta^k \xrightarrow{d} \beta$ on $D^J[0,1]$ as $k \to \infty$.*

It may be helpful for some readers to place the somewhat abstract development above on a more concrete footing. Toward that end, we examine a specific example where Assumption 3.1 holds. Suppose that $\Lambda_1, \ldots, \Lambda_J$ are J independent renewal processes with arrival rates λ_j, $1 \le j \le J$ and (finite) interarrival time variances v_j. For $k \ge 1$ and $0 \le t \le 1$, let

$$\Lambda_j^k(t) = \Lambda_j(kt). \tag{7}$$

(The speed-up of the kth arrival process by a factor of k in (7) corresponds to scaling up the arrival rate by a factor of k.) With $\Lambda_j^k(t)$ defined as in (7), Donsker's Theorem (Billingsley [1]) implies that Assumption 3.1 holds with $\sigma_{jj}^2 = \lambda_j^3 v_j$, $1 \le j \le J$, and $\sigma_{ij}^2 = 0$ for $i \ne j$.

The Skorohod Representation Theorem (cf. Whitt [14], Theorem 3.2.2) provides us with a way to replace the convergence in distribution of Assumption 3.1 by almost sure convergence, which simplifies our analysis. Applied in the context of Assumption 3.1, the Skorohod Representation Theorem states that there exist random elements $\tilde{\beta}^k$, $k \ge 1$, and $\tilde{\beta}$ defined on a common probability space such that

$$\tilde{\beta}^k \stackrel{d}{=} \beta^k, \qquad \tilde{\beta} \stackrel{d}{=} \beta, \tag{8}$$

and

$$P\Big(\lim_{k\to\infty} \tilde{\beta}^k = \tilde{\beta}\Big) = 1. \tag{9}$$

Rather than place the ˜ on all of our random variables, we employ a (reasonably standard) abuse of notation and simply assume that (9) holds for our original random elements. A consequence of (9) is that we can write, for $1 \le j \le J$,

$$\Lambda_j^k(t) = kt\lambda_j + \sqrt{k}\beta_j(t) + \epsilon_j^k(t), \tag{10}$$

where $\epsilon_j^k(t)$ satisfies

$$\lim_{k\to\infty} k^{-1/2} \sup_{0\le t\le 1} |\epsilon_j^k(t)| = 0 \quad \text{a.s.} \tag{11}$$

Recall that our performance measure is the expected total revenue. Weak (or almost sure) convergence does not guarantee convergence of expected values. We thus assume the following:

Assumption 3.2. *With $\epsilon_j^k(t)$ defined as in (10),*

$$\lim_{k\to\infty} k^{-1/2} E\Big[\sup_{0\le t\le 1} |\epsilon_j^k(t)|\Big] = 0, \quad 1 \le j \le J. \tag{12}$$

We do not provide general conditions under which Assumption 3.2 holds. We do, however, show that it holds for the independent renewal processes introduced above.

Lemma 3.1. *If $\Lambda_1, \ldots, \Lambda_J$ are independent renewal processes with finite second moment of interarrival times, and $\Lambda_j^k(t)$ is defined by (7) for $1 \le j \le J$, $0 \le t \le 1$, and $k \ge 1$, then Assumption 3.2 holds.*

See appendix for proof.

Our objective is to develop an admission policy that is asymptotically optimal on the diffusion scale, i.e.,

$$\lim_{k\to\infty} (E[\bar{R}^k] - E[R^k]) \big/ \sqrt{k} = 0, \tag{13}$$

where $E[\bar{R}^k]$ is the hindsight optimum and $E[R^k]$ is the expected revenue under our policy. To motivate the development of such a policy, in this section we first present Cooper's [3] result on fluid-scale asymptotic optimality of the booking limit policy that uses the solution of the fluid LP for booking limits. We then present an example showing that the more sensitive notion of asymptotic optimality represented by (13) cannot hold for any booking limit policy in the context of that example. We conjecture that this result holds in much greater generality, but we do not attempt a precise statement or a proof, both of which are beyond the scope of this paper. To indicate one of the issues that makes a precise delineation of the conditions under which (13) can hold for a booking limit policy, we show that, in fact, (13) does hold for a variant of our example.

3.1. Fluid scale asymptotic optimality of some existing schemes. Cooper [3] studies the asymptotic properties of the booking limit policy. Under the assumptions that customer arrivals follow some simple point processes, and scaling of the problem satisfies

$$C_l^k = kC_l \quad \text{and} \quad \Lambda_j^k(1)/k \xrightarrow{d} E[\Lambda_j(1)], \quad l = 1, 2, \ldots, L, \quad j = 1, 2, \ldots, J,$$

he proves that

$$\lim_{k\to\infty} \frac{\sum_j r_j X_j^k - E[R^k]}{k} = 0, \tag{14}$$

where $\{X_j^k, j = 1, 2, \ldots, J\}$ is the optimal solution of (2) for the kth problem and $E[R^k]$ is the expected revenue under the policy that uses X_j^k $(j = 1, 2, \ldots, J)$ as fixed booking limits. Since $\sum_j r_j X_j^k$ is the upper bound of the total revenue for any policy (Cooper [3]) including the hindsight optimum $E[\bar{R}^k]$, the above implies that

$$\lim_{k\to\infty} \frac{E[\bar{R}^k] - E[R^k]}{k} = 0. \tag{15}$$

In this paper, we use Equation (15) as the definition for asymptotic optimality on the fluid scale.

Talluri and van Ryzin [12] discuss the asymptotic properties of bid price control. They prove that if customer payment for the service is continuously distributed over a finite support, then the scheme is asymptotically optimal on the fluid scale. Furthermore, the difference from the hindsight optimum on the diffusion scale, i.e., the left-hand side of (13), converges to a constant (not necessarily zero) as k increases. They also show that if the price for the service takes discrete values, as is the case we are considering, the bid price control is not asymptotically optimal on the fluid scale.

3.2. Diffusion scale analysis of booking limit policies. Although many fixed booking limit policies are asymptotically optimal on the fluid scale, we provide a simple example where no fixed booking limit policy can be asymptotically optimal on the diffusion scale. The conclusion holds regardless of how the booking limits are set and even in cases where the use of booking limits is combined with nesting.

Consider the case where there is one resource with capacity C that is used in unit amount by two classes of customers. Suppose that $r_1 > r_2$; i.e., class 1 customers pay a higher price.

Assume that customer arrivals of the two classes are independent renewal processes with arrival rates λ_j $(j = 1, 2)$. We introduce a sequence of problems as specified by Equations (5)–(7). With just one resource $(L = 1)$ we do not need an index for it. Thus we let $C^k = kC$. We consider cases where

$$\lambda_1 + \lambda_2 > C \qquad \text{and} \qquad \lambda_1 < C; \tag{16}$$

i.e., the capacity is more than enough to serve the mean demand of class 1 customers, but is not sufficient to serve the combined mean demands of both classes. We assume that there is sufficient variation in the arrivals of class 1 customers in the sense that $\sigma_1^2 > 0$.

Define Π as the family of all fixed booking limit policy sequences. Each $p \in \Pi$ is associated with a sequence of booking limits $\{X_j^{k,p} (j = 1, 2),\ k \geq 1\}$ and satisfies the following conditions:

(i) Customers are accepted FCFS as long as admission is open to their class.

(ii) Admission to the low-paying class 2 customers closes when the number of accepted customers of that class hits $X_2^{k,p}$.

Note that the two conditions do not exclude policies that allow the high-paying class 1 customers to "borrow" or "steal" the booking limit from the low-paying class 2 customers. Therefore, the definition of Π is broad enough to include not only partitioned booking limits policies, but also nested ones (standard nesting as defined in Talluri and van Ryzin [13]).

Let $E[\bar{R}^k]$ be the hindsight optimum of the kth problem and let $E[R^{k,p}]$ be the expected revenue under any policy $p \in \Pi$. The conclusion we want to prove is formally stated as follows:

Theorem 3.1. *In the above problem instance, for any* $p \in \Pi$,

$$\lim_{k\to\infty} \frac{E[\bar{R}^k] - E[R^{k,p}]}{\sqrt{k}} > 0. \tag{17}$$

We let $W^k \in \mathcal{F}$, $k \geq 1$ denote a sequence of subsets of sample paths. By the definition of the hindsight optimum, $\bar{R}^k(\omega) \geq R^{k,p}(\omega)$ for all ω, including $\omega \notin W^k$. Therefore,

$$E[\bar{R}^k] - E[R^{k,p}] \geq E[\bar{R}^k 1(W^k)] - E[R^{k,p} 1(W^k)], \tag{18}$$

where $1(W^k)$ is the indicator function of W^k. This observation immediately leads to Lemma 3.2.

Lemma 3.2. *Let $\{W^k, k \geq 1\}$ be a sequence of sets with $W^k \in \mathcal{F}$ and $\lim_{k\to\infty} P(W^k) > 0$. For any policy $p \in \Pi$, if*

$$\lim_{k\to\infty} \frac{E[\bar{R}^k] - E[R^{k,p}]}{\sqrt{k}} = 0,$$

then

$$\lim_{k\to\infty} \inf_{\omega\in W^k} \frac{\bar{R}^k(\omega) - R^{k,p}(\omega)}{\sqrt{k}} = 0. \tag{19}$$

To prove Theorem 3.1, it is sufficient to show that no $p \in \Pi$ satisfies (19). We make this point in the following analysis of two cases, which leads to contradictory necessary conditions for the satisfaction of (19) by any $p \in \Pi$.

For the first case, let W^k be the collection of sample paths on which the number of class 1 customer arrivals falls below its mean by an amount of no less than $\sqrt{k}$, and the number of class 2 arrivals is at or above its mean, i.e., for $k \geq 1$,

$$\begin{aligned} W^k &= \{\Lambda_1^k(1) \leq k\lambda_1 - \sqrt{k} \text{ and } \Lambda_2^k(1) \geq k\lambda_2\} \\ &= \{\beta_1^k(1) \leq -1 \text{ and } \beta_2^k(1) \geq 0\}. \end{aligned} \tag{20}$$

By the assumption of renewal arrivals and the functional central limit theorem,

$$\lim_{k\to\infty} P(W^k) = P[\beta_1(1) \leq -1 \text{ and } \beta_2(1) \geq 0] > 0. \tag{21}$$

(Note that, if $\sigma_1^2 = 0$, the above limit is 0.) The revenue achieved under a policy $p \in \Pi$ satisfies

$$R^{k,p} \leq r_1 \Lambda_1^k(1) + r_2 X_2^{k,p}. \tag{22}$$

The hindsight optimal revenue satisfies (on W^k)

$$\begin{aligned} \bar{R}^k &= r_1\Lambda_1^k(1) + r_2[\Lambda_2^k(1) \wedge (C^k - \Lambda_1^k(1))] \\ &\geq r_1\Lambda_1^k(1) + r_2\big[k\lambda_2 \wedge (k(C-\lambda_1) + \sqrt{k})\big]. \end{aligned} \tag{23}$$

By the assumption that $\lambda_1 + \lambda_2 > C$, if k is sufficiently large,

$$k\lambda_2 \wedge \big(k(C-\lambda_1) + \sqrt{k}\big) = k(C-\lambda_1) + \sqrt{k},$$

and thus

$$\begin{aligned} \lim_{k\to\infty} \inf_{\omega\in W^k} \frac{\bar{R}^k(\omega) - R^{k,p}(\omega)}{\sqrt{k}} &\geq r_2 \lim_{k\to\infty} \frac{k(C-\lambda_1) + \sqrt{k} - X_2^{k,p}}{\sqrt{k}} \\ &= r_2 \lim_{k\to\infty}\left[1 - \frac{X_2^{k,p} - k(C-\lambda_1)}{\sqrt{k}}\right]. \end{aligned} \tag{24}$$

Therefore, to satisfy Lemma 3.2, we need

Necessary Condition 1.

$$\lim_{k\to\infty} \frac{X_2^{k,p} - k(C-\lambda_1)}{\sqrt{k}} \geq 1.$$

Now we deal with the second case. Let $t_0 = (C-\lambda_1)/\lambda_2$. Note by Equation (16) that $0 < t_0 < 1$. Consider cases where the number of arrivals of class 1 customers is below its mean by at least $\sqrt{k}$ in period $[0, t_0]$ and accelerates later in the process. In the same period $[0, t_0]$, the number of arrivals of class 2 customers is at least $\sqrt{k}$ more than its mean. Specifically, let

$$\begin{aligned} W^k &= \{\Lambda_1^k(t_0) \le k\lambda_1 t_0 - \sqrt{k},\ \Lambda_1^k(1) \ge k\lambda_1 \text{ and } \Lambda_2^k(t_0) \ge k\lambda_2 t_0 + \sqrt{k}\} \\ &= \{\beta_1^k(t_0) \le -1,\ \beta_1^k(1) \ge 0 \text{ and } \beta_2^k(t_0) \ge 1\}. \end{aligned} \tag{25}$$

The probability of this happening satisfies, by the functional central limit theorem,

$$\lim_{k\to\infty} P(W^k) = P[\beta_1(t_0) \le -1, \beta_1(1) \ge 0 \text{ and } \beta_2(t_0) \ge 1] > 0. \tag{26}$$

For each $\omega \in W^k$, the total number of customer arrivals satisfies

$$\Lambda_1^k(1) + \Lambda_2^k(1) \ge \Lambda_1^k(1) + \Lambda_2^k(t_0) \ge k\lambda_1 + (C^k - k\lambda_1) = C^k, \tag{27}$$

which is not less than the total capacity. Therefore, the hindsight optimum is achieved by accepting as many class 1 customers as possible and using the remaining capacity (if any) to serve class 2 customers. The resulting revenue satisfies

$$\bar{R} \ge r_2 C^k + (r_1 - r_2)[k\lambda_1 \wedge C^k] = r_2 C^k + (r_1 - r_2)k\lambda_1. \tag{28}$$

Under any policy $p \in \Pi$, the number of accepted class 2 customers in period $[0, t_0]$ is the minimum of (a) the number of class 2 customer arrivals; (b) the number allowed by capacity availability; and (c) the number allowed by the booking limit, which should be no less than

$$\begin{aligned} \Lambda_2^k(t_0) \wedge [C^k - \Lambda_1^k(t_0)] \wedge X_2^{k,p} &\ge (C^k - k\lambda_1 + \sqrt{k}) \wedge (C^k - k\lambda_1 t_0 + \sqrt{k}) \wedge X_2^{k,p} \\ &= (C^k - k\lambda_1 + \sqrt{k}) \wedge X_2^{k,p}. \end{aligned} \tag{29}$$

The number of class 1 customers served under policy p cannot exceed

$$C^k - [(C^k - k\lambda_1 + \sqrt{k}) \wedge X_2^{k,p}],$$

which equals $k\lambda_1 - \sqrt{k}$ if $X_2^{k,p} \ge C^k - k\lambda_1 + \sqrt{k}$. In this case, the revenue under policy p satisfies

$$R^{k,p} \le r_2 C^k + (r_1 - r_2)(k\lambda_1 - \sqrt{k}), \tag{30}$$

and consequently satisfies

$$\lim_{k\to\infty} \inf_{\omega \in W^k} \frac{\bar{R}^k(\omega) - R^{k,p}(\omega)}{\sqrt{k}} \ge (r_1 - r_2). \tag{31}$$

This means the satisfaction of (19) requires

Necessary Condition 2.

$$\lim_{k\to\infty} \frac{X_2^{k,p} - k(C - \lambda_1)}{\sqrt{k}} < 1.$$

The contradiction between Necessary Conditions 1 and 2 indicates that (19) in Lemma 3.2 cannot hold under any policy $p \in \Pi$, and thus proves Theorem 3.1. □

We observe that, if $\sigma_1^2 = 0$, there exist $X_j^{k,p}$ such that

$$\frac{E[\bar{R}^k] - E[R^{k,p}]}{\sqrt{k}} \to 0 \quad \text{as } k \to \infty, \tag{32}$$

which means there exist problem instances to which some fixed booking limit policy can be asymptotically optimal on the diffusion scale. The proof of (32) is given in the appendix.

4. An asymptotically optimal policy. In this section, we propose a new admission policy and analyze its asymptotic properties. We present the policy in §4.1 and prove that it is asymptotically optimal on the diffusion scale in §4.2.

4.1. Policy description. Our policy is called the Thinning and Trigger (T^2) policy and is composed of the following three elements:

(i) **Initial optimization and probabilistic acceptance rule.** At $t=0$, solve the LP

$$\max_{x_j}\left\{\sum_j r_j x_j \,\middle|\, \sum_j a_{lj}x_j \le C_l,\ 0\le x_j\le \lambda_j\right\}. \tag{33}$$

Let X_j $(j=1,2,\ldots,J)$ denote a solution. We then accept class j customers with probability X_j/λ_j. Based on this rule, customers of class j are all accepted if $X_j=\lambda_j$, all rejected if $X_j=0$, and partially accepted if $0<X_j<\lambda_j$. Correspondingly, we divide customer classes into three groups: $j\in\mathcal{J}_\lambda$ if $X_j=\lambda_j$, $j\in\mathcal{J}_0$ if $X_j=0$, and $j\in\mathcal{J}_x$ if $0<X_j<\lambda_j$.

(ii) **Trigger function and thresholds.** Let $z_j(t)$ denote the number of class j customers accepted by time t under the above probabilistic acceptance rule. Define $\Upsilon_j(t)=z_j(t)-X_jt$ $(j=1,2,\ldots,J)$ as the difference between the actual number accepted and the expected acceptance at time t. The trigger function is defined to be

$$\Gamma(t)=\alpha\sum_{j\in\mathcal{J}_\lambda}|\Upsilon_j(t)|=\alpha\sum_{j\in\mathcal{J}_\lambda}|z_j(t)-\lambda_jt|, \tag{34}$$

where α is a constant whose value is determined as follows: consider the incidence matrix $A=(a_{lj})$, $l=1,2,\ldots,L$, $j=1,2,\ldots,J$. For each nonsingular submatrix of A, pick the maximum absolute value of all elements in its inverse. Take the maximum of all these values over all nonsingular submatrices. If the resulting value is greater than 1, set α to that value. Otherwise, set $\alpha=1$.

We explain the rationale behind the value of α in the proof of Theorem 4.2. It will also become evident that, for the ease of calculation, we can replace α with a larger constant in the trigger function. For example, in case each resource is used in unit amount (i.e., a_{lj} is either 1 or 0 for all $j=1,2,\ldots,J$ and $l=1,2,\ldots,L$), then $\alpha\le\max[1,L\wedge J-1]$. In this case, we can define the trigger function as

$$\Gamma(t)=\max[1,L\wedge J-1]\sum_{j\in\mathcal{J}_\lambda}|\Upsilon_j(t)|,$$

which yields the same asymptotic results.

Let $\underline{\theta}_j(t)\equiv(1-t)X_j$ and $\bar{\theta}_j(t)\equiv(1-t)(\lambda_j-X_j)$, and define

$$\begin{aligned} s_j(t)&=\underline{\theta}_j(t)-|\Upsilon_j(t)| \quad \forall\, j\colon X_j>0,\\ S_j(t)&=\bar{\theta}_j(t)-|\Upsilon_j(t)| \quad \forall\, j\colon X_j<\lambda_j \end{aligned} \tag{35}$$

as two sets of thresholds.

(iii) **Reoptimization**. At $t=0$, $\Gamma(0)=0$, $s_j(0)>0$, and $S_j(0)>0$. These values evolve continuously over time and are compared whenever one decides whether to accept or reject a customer. Let τ be the first time that the comparison shows that $\Gamma(t)$ exceeds a threshold value, i.e.,

$$\tau=\inf_{t\ge0}\left\{\Gamma(t)\ge\min_{j\in\mathcal{J}_\lambda\cup\mathcal{J}_x}s_j(t)\wedge\min_{j\in\mathcal{J}_0\cup\mathcal{J}_x}S_j(t)\right\}\wedge1. \tag{36}$$

At this point, recalculate the acceptance ratio by solving the LP

$$\max_{x_j}\left\{\sum_j r_jx_j \,\middle|\, \sum_j a_{lj}x_j\le C_l(\tau),\ 0\le x_j\le(1-\tau)\lambda_j\right\}, \tag{37}$$

where $C_l(\tau)\equiv C_l-\sum_j a_{lj}z_j(\tau)$ is the amount of capacity of resource l left at τ, and $(1-\tau)\lambda_j$ is a proxy for the expected number of class j customer arrivals in the remaining period $(\tau,1]$. Denote the solution of the reoptimization by $X_j(\tau)$. The new probability for acceptance is $X_j(\tau)/\lambda_j(1-\tau)$ and applies all the way to the end of the time interval, at $t=1$. (In other words, no more reoptimization is necessary.)

4.2. Asymptotic properties of the T^2 policy. We now analyze each element of the T^2 policy and prove that it is asymptotically optimal on the diffusion scale.

4.2.1. Probabilistic acceptance rule. For $k=1,2,\ldots,\lambda_j^k=k\lambda_j$, $j=1,2,\ldots,J$, and $C_l^k=kC_l$, $l=1,2,\ldots,L$, so $X_j^k=kX_j$. Therefore, the same acceptance ratios X_j/λ_j apply to problems of all sizes.

In the kth problem, imposing the probabilistic acceptance rule on the class j ($j=1,2,\ldots,J$) arrival process

$$\Lambda_j^k(t)=k\lambda_j t+\sqrt{k}\beta_j(t)+\epsilon_j^k(t), \quad 0\le t\le 1, \tag{38}$$

induces a new process z_j^k in the following manner: For each j, $1\le j\le J$, let $\{\psi_{j,i}, i\ge 1\}$ be a sequence of i.i.d. random variables with $P\{\psi_{j,i}=1\}=1-P\{\psi_{j,i}=0\}=X_j/\lambda_j$, with all J of these sequences independent. Then

$$z_j^k(t)=\sum_{i=1}^{\Lambda_j^k(t)}\psi_{j,i}.$$

For $j=1,2,\ldots,J$, let

$$\phi_j^k(t)=\frac{\sum_{i=1}^{\lceil kt\rceil}\psi_{j,i}-ktX_j/\lambda_j}{\sqrt{k}}.$$

Then $\phi_j^k\to^d\phi_j$ where ϕ_j is a driftless Brownian motion with infinitesimal variance $\sigma_j^2\equiv X_j(\lambda_j-X_j)/\lambda_j^2$. Note that $\sigma_j^2>0$ if and only if $j\in\mathcal{J}_x$. We can then write

$$z_j^k(t)=\frac{X_j}{\lambda_j}\Lambda_j^k(t)+\sqrt{k}\phi_j^k\left(\frac{\Lambda_j^k(t)}{k}\right).$$

We apply the Skorohod Representation Theorem to the sequence of processes $\{\phi_j^k, k\ge 1\}$, utilizing the same abuse of notation as before, which allows us to write

$$\phi_j^k(t)=\phi_j(t)+\frac{\delta_j^k(t)}{\sqrt{k}},$$

with $k^{-1/2}\sup_{0\le t\le T}|\delta_j^k(t)|\to 0$ a.s. as $k\to\infty$ for any $T<\infty$. Thus we can write, for $0\le t\le 1$,

$$z_j^k(t)=kX_jt+\sqrt{k}\frac{X_j}{\lambda_j}\beta_j(t)+\frac{X_j}{\lambda_j}\epsilon_j^k(t)+\sqrt{k}\phi_j\left(\frac{\Lambda_j^k(t)}{k}\right)+\delta_j^k\left(\frac{\Lambda_j^k(t)}{k}\right). \tag{39}$$

Under Assumption 3.1 and the Skorohod Representation Theorem,

$$\sup_{0\le t\le 1}\left|\frac{\Lambda_j^k(t)}{k}-\lambda_jt\right|\to 0 \quad \text{a.s. as } k\to\infty,$$

so the almost sure continuity of Brownian motion yields

$$\sup_{0\le t\le 1}\left|\phi_j\left(\frac{\Lambda_j^k(t)}{k}\right)-\phi_j(\lambda_jt)\right|\to 0 \quad \text{a.s. as } k\to\infty,$$

and

$$k^{-1/2}\sup_{0\le t\le 1}\left|\delta_j^k\left(\frac{\Lambda_j^k(t)}{k}\right)\right|\to 0 \quad \text{a.s. as } k\to\infty.$$

Denote by τ^k the time that the trigger value first reaches a threshold and invokes reoptimization in the kth problem. The following lemma provides bounds and limits needed in our proofs.

Lemma 4.1. *If Assumptions 3.1 and 3.2 hold, then*

$$\text{(a)}\quad \sup_{k\ge 1}E\left[\left(\phi_j\left(\frac{\Lambda_j^k(t)}{k}\right)\right)^2\right]<\infty, \quad 0\le t\le 1, \tag{40}$$

$$\text{(b)}\quad \sup_{k\ge 1}E\left[\left(\phi_j\left(\frac{\Lambda_j^k(\tau^k)}{k}\right)\right)^2\right]<\infty, \tag{41}$$

$$\text{(c)}\quad \lim_{k\to\infty}E\left[\left|\delta_j^k\left(\frac{\Lambda_j^k(t)}{k}\right)\right|\right]=0, \quad 0\le t\le 1, \tag{42}$$

$$\text{(d)}\quad \lim_{k\to\infty}E\left[\left|\delta_j^k\left(\frac{\Lambda_j^k(\tau^k)}{k}\right)\right|\right]=0. \tag{43}$$

See appendix for proof.

Theorem 4.1 shows that the probabilistic acceptance rule alone is sufficient to achieve asymptotic optimality on the fluid scale.

THEOREM 4.1 *Let $\{X_j,\ 1 \le j \le J\}$ be a solution to the fluid LP (33). Suppose class j customers are accepted with probability X_j/λ_j until the capacity of a required resource runs out, $1 \le j \le J$. Let $E[R^k]$ be the resulting expected revenue and let $E[\bar{R}^k]$ be the hindsight optimum in the kth problem. Under the scaling (5), if Assumptions 3.1 and 3.2 hold, then*

$$\lim_{k\to\infty} \frac{E[\bar{R}^k] - E[R^k]}{k} = 0.$$

See appendix for proof.

To extend asymptotic optimality from the fluid scale to the diffusion scale requires the trigger-and-threshold mechanism, the properties of which we discuss next.

4.2.2. Trigger function and threshold. Let $\Gamma^k(1)$ be the trigger value at $t = 1$ of the kth problem (without reoptimization); i.e.,

$$\Gamma^k(1) = \alpha \sum_{j\in\mathcal{J}_\lambda} |z_j^k(1) - k\lambda_j| = \alpha \sum_{j\in\mathcal{J}_\lambda} |\Lambda_j^k(1) - k\lambda_j|.$$

The value is used in Theorem 4.2 to bound the hindsight optimal solution.

THEOREM 4.2. *Let $\{X_j, 1 \le j \le J\}$ be an optimal solution to the fluid LP (33). Under the scaling (5), if Assumption 3.1 holds, then a.s. for large enough k, there exist*

$$\bar{z}_j^k \in [kX_j - \Gamma^k(1),\ kX_j + \Gamma^k(1)], \quad j = 1, 2, \ldots, J, \tag{44}$$

where $\{\bar{z}_j^k,\ 1 \le j \le J\}$ solves the hindsight LP

$$\max_{x_j} \left\{ \sum_j r_j x_j \,\middle|\, \sum_j a_{lj} x_j \le kC_l,\ 0 \le x_j \le \Lambda_j^k(1) \right\}. \tag{45}$$

PROOF. Define $\eta_j^k \equiv \Lambda_j^k(1) - k\lambda_j = \sqrt{k}\beta_j(1) + \epsilon_j^k(1)$, so that $\Gamma^k(1) = \alpha \sum_{j\in\mathcal{J}_\lambda} |\eta_j^k|$. By Assumption 3.1, a.s. for large enough k,

$$|\eta_j^k| \le k(\lambda_j - X_j), \quad \text{for } j \notin \mathcal{J}_\lambda. \tag{46}$$

Let y_j^k be a hindsight optimal solution that optimizes

$$\max_{\bar{x}_j} \left\{ \sum_{j=1}^J r_j x_j \,\middle|\, \sum_{j=1}^J a_{lj} x_j \le kC_l,\ 0 \le x_j \le k\lambda_j + \eta_j^k \right\}. \tag{47}$$

Define $\mathcal{L}_t = \{l \in \{1, \ldots, L\}\colon \sum_{j=1}^J a_{lj} X_j = C_l\}$. By definition and (46),

$$\sum_{j=1}^J a_{lj} kX_j \le kC_l \quad l = 1, \ldots, L, \tag{48}$$

$$-\sum_{j=1}^J a_{lj} kX_j \le -\sum_{j=1}^J a_{lj} y_j^k \quad l \in \mathcal{L}_t, \tag{49}$$

$$0 \le kX_j \le k\lambda_j + \eta_j^k \quad j \notin \mathcal{J}_\lambda, \tag{50}$$

$$0 \le kX_j \le k\lambda_j \quad j \in \mathcal{J}_\lambda, \tag{51}$$

$$-kX_j \le -y_j^k + \eta_j^k \quad j \in \mathcal{J}_\lambda, \tag{52}$$

$$kX_j \le y_j^k \quad j \in \mathcal{J}_0. \tag{53}$$

Denote the parameter matrix on the left-hand side of the above by

$$B = \begin{pmatrix} A \\ -A_t \\ N \end{pmatrix}, \tag{54}$$

where $A = (a_{lj})$, A_t is a submatrix of A containing only rows corresponding to $l \in \mathcal{L}_t$, and each row in submatrix N is either $\vec{0}$ or $\pm\vec{e}$. Let α be the minimum value such that there is no nonsingular submatrix of B whose inverse contains an entry that is larger than α in the absolute value. Given the structure of B, α is either 1 or the upper bound on the absolute value of all entries in the inverse of any nonsingular submatrix of A, whichever is larger.

We now show that there exist

$$\bar{z}_j^k \in \left[kX_j - \alpha \sum_{j' \in \mathcal{J}_\lambda} |\eta_{j'}^k|, kX_j + \alpha \sum_{j' \in \mathcal{J}_\lambda} |\eta_{j'}^k|\right] \quad j = 1, 2, \ldots, J,$$

such that

$$\sum_{j=1}^{J} a_{lj}\bar{z}_j^k \le kC_l \quad l = 1, \ldots, L, \tag{55}$$

$$-\sum_{j=1}^{J} a_{lj}\bar{z}_j^k \le -\sum_{j=1}^{J} a_{lj}y_j^k \quad l \in \mathcal{L}_t, \tag{56}$$

$$0 \le \bar{z}_j^k \le k\lambda_j + \eta_j^k \quad j \notin \mathcal{J}_\lambda, \tag{57}$$

$$0 \le \bar{z}_j^k \le k\lambda_j + \eta_j^k \quad j \in \mathcal{J}_\lambda, \tag{58}$$

$$-\bar{z}_j^k \le -y_j^k \quad j \in \mathcal{J}_\lambda, \tag{59}$$

$$\bar{z}_j^k \le y_j^k \quad j \in \mathcal{J}_0. \tag{60}$$

This existence result is proved by applying the proof of Theorem 10.5 in Schrijver [10] with a slight change: comparing the right-hand side of constraints (55)–(60) and constraints (48)–(53), the perturbations η_j^k $(j \in \mathcal{J}_\lambda)$ in Equation (58) and $-\eta_j^k$ $(j \in \mathcal{J}_\lambda)$ in Equation (59) correspond to nonzero elements of $b'' - b'$ in Theorem 10.5 of Schrijver [10]. We want a version of Theorem 10.5 in Schrijver [10] with the right-hand side of their Equation (25) ($n\Delta\|b' - b''\|_\infty$ in their notation) replaced by (in our notation) $\Gamma^k(1) = \alpha \sum_{j \in \mathcal{J}_\lambda} |\eta_j^k|$. Let $\epsilon = \Gamma^k(1)$. Our desired result follows if we replace the inequality

$$y(b'' - b') + \epsilon \ge -\|y\|_1 \|b' - b''\|_\infty + \epsilon \ge 0$$

in Equation (27) of their proof with

$$y(b'' - b') + \epsilon \ge -\|y\|_\infty \sum_{j \in \mathcal{J}_\lambda} |\eta_j^k| + \epsilon \ge 0.$$

Note that $\bar{z}_j^k$ $(j = 1, 2, \ldots, J)$ is feasible for (47). To show it is also optimal, expanding the second part of the proof of Theorem 10.5 in Schrijver [10] with details of our model, let μ_l $(l = 1, 2, \ldots, L)$ and γ_j $(j = 1, 2, \ldots, J)$ be the Lagrange multipliers associated with the constraints $\sum_{j=1}^{J} a_{lj}x_j \le C_l$ and $x_j \le \lambda_j$ in the fluid LP (2), respectively. Observe that

$$\mu_l = 0 \quad \text{if } l \notin \mathcal{L}_t \left(\text{i.e.,} \sum_{j=1}^{J} a_{lj}x_j < C_l\right),$$

$$\gamma_j \ge (=, \le) 0 \quad \text{if } j \in \mathcal{J}_\lambda(\mathcal{J}_x, \mathcal{J}_0), \qquad \text{and}$$

$$r_j = \gamma_j + \sum_{l=1}^{L} a_{lj}\mu_l \quad \text{if } 1 \le j \le J.$$

Multiply both sides of (56) by the corresponding μ_l, both sides of (58) and (59) by the corresponding γ_j, sum the products, and apply the above conditions. Then

$$\sum_{j=1}^{J} r_j\bar{z}_j^k = \sum_{j=1}^{J} \gamma_j\bar{z}_j^k + \sum_{l \in \mathcal{L}_t} \mu_l \sum_{j=1}^{J} a_{lj}\bar{z}_j^k \ge \sum_{j=1}^{J} \gamma_j y_j^k + \sum_{l \in \mathcal{L}_t} \mu_l \sum_{j=1}^{J} a_{lj}y_j^k = \sum_{j=1}^{J} r_j y_j^k. \tag{61}$$

The conclusion is immediate by the definition of y_j^k $(j = 1, 2, \ldots, J)$. $\square$

The trigger mechanism starts to affect the admission process at τ^k after the trigger value first reaches a threshold. Let $z_j^k(\tau^k)$ be the number of class j customers that have been accepted by that time. At this point, the final number of accepted class j customers will be in the range

$$[z_j^k(\tau^k), z_j^k(\tau^k)+\Lambda_j^k(1)-\Lambda_j^k(\tau^k)] = [z_j^k(\tau^k), z_j^k(\tau^k)+k\lambda_j(1-\tau^k)+\Delta_j^k(\tau^k)], \tag{62}$$

where

$$\Delta_j^k(\tau^k) \equiv \sqrt{k}[\beta_j(1)-\beta_j(\tau^k)]+\epsilon_j^k(1)-\epsilon_j^k(\tau^k)$$

is the deviation of the actual number of class j arrivals from its mean in $(\tau^k, 1]$. The lower bound is reached when no class j customer is accepted after τ^k, and the upper bound is reached when all of them are accepted.

For all $t<\tau^k$ and for all j such that $X_j>0$,

$$kX_j - z_j^k(t) = \underline{\theta}_j^k(t) - \Upsilon_j^k(t) \geq \underline{\theta}_j^k(t) - |\Upsilon_j^k(t)| > \Gamma^k(t) \geq 0, \tag{63}$$

so that

$$\sum_{j=1}^J a_{lj} z_j^k(t) \leq \sum_{j=1}^J a_{lj}(kX_j) \leq kC_l, \quad l=1,2,\ldots,L.$$

Consequently, before τ^k, capacity constraints do not interfere with the admission, so, according to (39),

$$z_j^k(\tau^k-) = kX_j\tau^k + \sqrt{k}\frac{X_j}{\lambda_j}\beta_j(\tau^k-) + \frac{X_j}{\lambda_j}\epsilon_j^k(\tau^k-) + \sqrt{k}\phi_j\left(\frac{\Lambda_j^k(\tau^k-)}{k}\right) + \delta_j^k\left(\frac{\Lambda_j^k(\tau^k-)}{k}\right), \quad j=1,2,\ldots,J.$$

Based on the above discussions and Theorem 4.2, the following theorem establishes a bound between the hindsight optimum and the best solution one can find in the range of (62).

Theorem 4.3. *There exist*

$$\tilde{z}_j^k \in [z_j^k(\tau^k), z_j^k(\tau^k)+k\lambda_j(1-\tau^k)+\Delta_j^k(\tau^k)], \quad 1\leq j\leq J,$$

such that $\sum_{j=1}^J a_{lj}\tilde{z}_j^k \leq kC_l$, $l=1,2,\ldots,L$ *and*

$$\sum_{j=1}^J r_j \tilde{z}_j^k \geq \bar{R}^k - \sum_{j=1}^J r_j[|\Delta_j^k(\tau^k)| + G|\Gamma^k(1)-\Gamma^k(\tau^k)|], \tag{64}$$

where $G = |J|\max_l\{\max_{j,j'}\{a_{lj}/a_{lj'} \mid a_{lj'}>0\}\}$ *is a constant.*

Proof. Following Theorem 4.2, let $\bar{z}_j^k$ $(j=1,2,\ldots,J)$ be a hindsight optimal solution such that

$$kX_j - \Gamma^k(1) \leq \bar{z}_j^k \leq kX_j + \Gamma^k(1).$$

Define $\mathcal{J}_1 = \{j \mid \bar{z}_j^k < z_j^k(\tau^k),\ j=1,2,\ldots,J\}$. It follows that $|\mathcal{J}_1|<J$ and $X_j>0$ if $j\in\mathcal{J}_1$. Define

$$\Psi = \left(\max_{l,j} a_{lj}\right)|\mathcal{J}_1||\Gamma^k(1)-\Gamma^k(\tau^k)|.$$

Recall that $\Upsilon_j^k(t)\equiv z_j^k(t)-kX_jt$, and

$$\underline{\theta}_j^k(t) \equiv kX_j(1-t) \quad (X_j>0), \qquad \bar{\theta}_j^k(t) \equiv k(\lambda_j-X_j)(1-t) \quad (X_j<\lambda_j). \tag{65}$$

Also, by the definition of τ^k, for all $t<\tau^k$,

$$-\Gamma^k(t) \geq -\underline{\theta}_j^k(t) + |\Upsilon_j^k(t)| \quad (X_j>0) \qquad \text{and} \qquad \Gamma^k(t) \leq \bar{\theta}_j^k(t) - |\Upsilon_j^k(t)| \quad (X_j<\lambda_j). \tag{66}$$

(i) If $j \in \mathcal{J}_1$, let $\tilde{z}_j^k = z_j^k(\tau^k)$. For all $l \in \mathcal{L}$,

$$\begin{aligned}\sum_{j\in\mathcal{J}_1} a_{lj}(\tilde{z}_j^k - \bar{z}_j^k) &\le \sum_{j\in\mathcal{J}_1} a_{lj}[z_j^k(\tau^k) - (kX_j - \Gamma^k(1))]\\ &= \sum_{j\in\mathcal{J}_1} a_{lj}[\Upsilon_j^k(\tau^k) - \underline{\theta}_j^k(\tau^k) + \Gamma^k(1)]\\ &\le \sum_{j\in\mathcal{J}_1} a_{lj}[-\Gamma^k(\tau^k) + \Gamma^k(1)] \le \Psi. \end{aligned} \tag{67}$$

(ii) If $z_j^k(\tau^k) \le \bar{z}_j^k \le z_j^k(\tau^k) + k\lambda_j(1-\tau^k) + \Delta_j^k(\tau^k)$, let

$$\tilde{z}_j^k = \max\{z_j^k(\tau^k),\ \bar{z}_j^k - \Psi/a_{j\min}\}, \qquad a_{j\min} \equiv \min_l\{a_{lj} \mid a_{lj} > 0\}.$$

(iii) If $\bar{z}_j^k > z_j^k(\tau^k) + k\lambda_j(1-\tau^k) + \Delta_j^k(\tau^k)$, let

$$\tilde{z}_j^k = z_j^k(\tau^k) + [k\lambda_j(1-\tau^k) + \Delta_j^k(\tau^k) - \Psi/a_{j\min}]^+.$$

Then

$$\begin{aligned}\bar{z}_j^k - \tilde{z}_j^k &\le kX_j + \Gamma^k(1) - \tilde{z}_j^k\\ &\le \Gamma^k(1) - [\Upsilon_j^k(\tau^k) + \bar{\theta}_j^k(\tau^k) + \Delta_j^k(\tau^k) - \Psi/a_{j\min}]\\ &\le |\Gamma^k(1) - \Gamma^k(\tau^k)| + |\Delta_j^k(\tau^k)| + \Psi/a_{j\min}\\ &\le |\Delta_j^k(\tau^k)| + G|\Gamma^k(1) - \Gamma^k(\tau^k)|. \end{aligned} \tag{68}$$

Under the above construction,

$$\begin{aligned}\tilde{z}_j^k &\in [z_j^k(\tau^k), z_j^k(\tau^k) + k\lambda_j(1-\tau^k) + \Delta_j^k(\tau^k)],\\ \tilde{z}_j^k &\ge \bar{z}_j^k - |\Delta_j^k(\tau^k)| - G|\Gamma^k(1) - \Gamma^k(\tau^k)|. \end{aligned} \tag{69}$$

To complete the proof, we need to show that

$$\sum_{j=1}^J a_{lj}\tilde{z}_j^k \le kC_l, \quad l = 1, 2, \ldots, L. \tag{70}$$

It follows from the above construction that, for $j \notin \mathcal{J}_1$,

$$\tilde{z}_j^k \le \max\{z_j^k(\tau^k), \bar{z}_j^k - \Psi/a_{j\min}\}.$$

Define $\mathcal{J}_2 = \{j \notin \mathcal{J}_1\colon \tilde{z}_j^k = z_j^k(\tau^k)\}$ and $\mathcal{J}_3 = \{j \notin \mathcal{J}_1\colon z_j^k(\tau^k) < \tilde{z}_j^k\}$.

If $\mathcal{J}_3 = \varnothing$, then

$$\sum_{j\notin\mathcal{J}_1} a_{lj}\tilde{z}_j^k = \sum_{j\notin\mathcal{J}_1} a_{lj} z_j^k(\tau^k),$$

so that

$$\sum_{j=1}^J a_{lj}\tilde{z}_j^k = \sum_{j=1}^J a_{lj} z_j^k(\tau^k) \le kC_l.$$

If $\mathcal{J}_3 \ne \varnothing$, then

$$\sum_{j\notin\mathcal{J}_1} a_{lj}\tilde{z}_j^k \le \sum_{j\in\mathcal{J}_2} a_{lj}\bar{z}_j^k + \sum_{j\in\mathcal{J}_3} a_{lj}(\bar{z}_j^k - \Psi/a_{j\min}), \tag{71}$$

so that, combining (67) and (71),

$$\sum_{j=1}^J a_{lj}\tilde{z}_j^k \le \sum_{j\in\mathcal{J}_1} a_{lj}\bar{z}_j^k + \Psi + \sum_{j\in\mathcal{J}_2} a_{lj}\bar{z}_j^k + \sum_{j\in\mathcal{J}_3} a_{lj}\bar{z}_j^k - \Psi\sum_{j\in\mathcal{J}_3}\frac{a_{lj}}{a_{j\min}} \le \sum_{j=1}^J a_{lj}\bar{z}_j^k \le kC_l.$$

Note that in (71) we used $\bar{z}_j^k \ge z_j^k(\tau^k)$ for $j \notin \mathcal{J}_1$. $\square$

As an immediate implication of Theorem 4.3, let $\bar{R}^k(\tau^k, 1)$ be the hindsight optimal revenue for the period $(\tau^k, 1]$ after $z_j^k(\tau^k)$ $(j = 1, 2, \ldots, J)$ customers have already been accepted. Then

$$\bar{R}^k - \left(\sum_{j=1}^J r_j z_j^k(\tau^k) + \bar{R}^k(\tau^k, 1)\right) \le \sum_{j=1}^J r_j[|\Delta_j^k(\tau^k)| + G|\Gamma^k(1) - \Gamma^k(\tau^k)|]. \tag{72}$$

Our next theorem shows that this difference converges to zero on the diffusion scale.

Theorem 4.4. *Given τ^k as defined above, under the scaling (5), if Assumptions 3.1 and 3.2 hold, there exists a constant $M < \infty$ such that*

$$0 \le \lim_{k\to\infty} E\big[\sqrt{k}(1-\tau^k)\big] \le M, \tag{73}$$

which implies that

$$\lim_{k\to\infty} E[1-\tau^k] = 0. \tag{74}$$

As a result,

$$\lim_{k\to\infty} \frac{E[|\Delta_j^k(\tau^k)| + |\Gamma^k(1) - \Gamma^k(\tau^k)|]}{\sqrt{k}} = 0, \quad j = 1, 2, \ldots, J. \tag{75}$$

Proof. By definition of τ^k, either

$$\Gamma^k(\tau^k) \ge \underline{\theta}_j^k(\tau^k) - |\Upsilon_{j'}^k(\tau^k)| \quad \text{for some } j' \notin \mathcal{J}_0, \quad \text{or} \tag{76}$$

$$\Gamma^k(\tau^k) \ge \bar{\theta}_j^k(\tau^k) - |\Upsilon_{j'}^k(\tau^k)| \quad \text{for some } j' \notin \mathcal{J}_\lambda. \tag{77}$$

If (76) is true,

$$\begin{aligned}\alpha \sum_{j\in\mathcal{J}_\lambda} &|\sqrt{k}\beta_j(\tau^k) + \epsilon_j^k(\tau^k)| \\ &\ge kX_{j'}(1-\tau^k) - \left|\sqrt{k}\frac{X_{j'}}{\lambda_{j'}}\beta_{j'}(\tau^k) + \frac{X_{j'}}{\lambda_{j'}}\epsilon_{j'}^k(\tau^k) + \sqrt{k}\phi_{j'}\left(\frac{\Lambda_{j'}^k(\tau^k)}{k}\right) + \delta_{j'}^k\left(\frac{\Lambda_{j'}^k(\tau^k)}{k}\right)\right|.\end{aligned} \tag{78}$$

Thus

$$\begin{aligned}\sqrt{k}(1-\tau^k) \le &\left[\alpha\sum_{j\in\mathcal{J}_\lambda}|\beta_j(\tau^k)| + \frac{X_{j'}}{\lambda_{j'}}|\beta_{j'}(\tau^k)| + \left|\phi_{j'}\left(\frac{\Lambda_{j'}^k(\tau^k)}{k}\right)\right|\right]\Big/ X_{j'} \\ &+ \left[\alpha\sum_{j\in\mathcal{J}_\lambda}|\epsilon_j^k(\tau^k)| + \frac{X_{j'}}{\lambda_{j'}}|\epsilon_{j'}^k(\tau^k)| + \left|\delta_{j'}^k\left(\frac{\Lambda_{j'}^k(\tau^k)}{k}\right)\right|\right]\Big/(\sqrt{k}X_{j'}).\end{aligned} \tag{79}$$

For $1 \le j \le J$, as $k \to \infty$,

$$E[|\epsilon_j^k(\tau^k)|]/\sqrt{k} \to 0 \qquad \text{and} \qquad E\left[\left|\delta_j^k\left(\frac{\Lambda_j^k(\tau^k)}{k}\right)\right|\right]\Big/\sqrt{k} \to 0$$

by Assumption 3.2 and Lemma 4.1, respectively. Taking expectations on both sides and letting $k \to \infty$, we obtain, again using Lemma 4.1,

$$\lim_{k\to\infty} E[\sqrt{k}(1-\tau^k)] \le \left(\alpha\sum_{j\in\mathcal{J}_\lambda} E[|\beta_j(\tau^k)|] + \frac{X_{j'}}{\lambda_{j'}}E[|\beta_{j'}(\tau^k)|] + E\left[\left|\phi_{j'}\left(\frac{\Lambda_{j'}^k(\tau^k)}{k}\right)\right|\right]\right)\Big/X_{j'} \equiv M.$$

A similar procedure shows that the conclusion applies if (77) is true.

As for the second part, recall that

$$\Delta_j^k(\tau^k) = \sqrt{k}\big[\beta_j(1) - \beta_j(\tau^k)\big] + \epsilon_j^k(1) - \epsilon_j^k(\tau^k)$$

$$|\Gamma^k(1) - \Gamma^k(\tau^k)| = \alpha\left|\sum_{j\in\mathcal{J}_\lambda}\big[|\sqrt{k}\beta_j(1) + \epsilon_j^k(1)| - |\sqrt{k}\beta_j(\tau^k) + \epsilon_j^k(\tau^k)|\big]\right|.$$

Under Assumption 3.2, it suffices to show that

$$\lim_{k\to\infty} E[|\beta_j(1) - \beta_j(\tau^k)|] = 0. \tag{80}$$

Note that

$$\frac{\beta_j(\tau^k,1)}{1-\tau^k} \equiv \frac{\beta_j(1)-\beta_j(\tau^k)}{1-\tau^k}$$

is normally distributed with zero mean and a finite standard deviation. By Schwarz's inequality,

$$E[\beta_j(\tau^k,1)] = E\left(|1-\tau^k|\left|\frac{\beta_j(\tau^k,1)}{1-\tau^k}\right|\right) \le \sqrt{E[(1-\tau^k)^2]}\sqrt{E\left[\left(\frac{\beta_j(\tau^k,1)}{1-\tau^k}\right)^2\right]}. \tag{81}$$

Since $\tau^k \le 1$, $(1-\tau^k)^2 \le 1-\tau^k$, so that using (81), (80) follows from (74) and yields (75). □

We would be able to conclude that our policy is asymptotically optimal on the diffusion scale if the reoptimization invoked by the trigger mechanism could deliver a revenue of $\bar{R}^k(\tau^k,1)$ after τ^k. Even though this cannot be the case, we show next that the difference is sufficiently small that the conclusion is still valid.

4.2.3. Reoptimization. We divide the revenue generated under the T^2 policy (R^k) into two parts, $\sum_{j=1}^J r_j z_j^k(\tau^k)$, which is collected before the trigger point, and $R^k(\tau^k,1)$, revenue obtained afterward. It follows that

$$R^k = \sum_{j=1}^J r_j z_j^k(\tau^k) + R^k(\tau^k,1) = \left[\sum_{j=1}^J r_j z_j^k(\tau^k) + \bar{R}^k(\tau^k,1)\right] - [\bar{R}^k(\tau^k,1) - R^k(\tau^k,1)]. \tag{82}$$

Take the expectation of the above and note that by Theorem 4.4 and (72),

$$\left(E[\bar{R}^k] - E\left[\sum_{j=1}^J r_j z_j^k(\tau^k) + \bar{R}^k(\tau^k,1)\right]\right)\Big/\sqrt{k} \to 0 \quad \text{as } k\to\infty. \tag{83}$$

To prove that the T^2 policy is asymptotically optimal on the diffusion scale, we only need to show that $E[R^k(\tau^k,1)]$ differs from $E[\bar{R}^k(\tau^k,1)]$ by $o(\sqrt{k})$. Intuitively, $1-\tau^k$ is on the order of $1/\sqrt{k}$ by Theorem 4.4, so $E[\bar{R}^k(\tau^k,1)]$ should be on the order of $\sqrt{k}$. We also know from Theorem 4.1 that the probabilistic acceptance rule is asymptotically optimal on the fluid scale. Combining these two facts leads us to conclude that the difference between $E[R^k(\tau^k,1)]$ and $E[\bar{R}^k(\tau^k,1)]$ should be of a lower order than $\sqrt{k}$. This conclusion is formally stated as the following theorem, which also formally states our main result.

Theorem 4.5. *Given τ^k as defined above, under the scaling (5), if Assumptions 3.1 and 3.2 hold, then*

$$\lim_{k\to\infty} \frac{E[\bar{R}^k(\tau^k,1) - R^k(\tau^k,1)]}{\sqrt{k}} = 0, \tag{84}$$

so that

$$\lim_{k\to\infty} \frac{E[\bar{R}^k - R^k]}{\sqrt{k}} = 0. \tag{85}$$

Proof. Apply the same reasoning that leads to (105) for the acceptance process over $[\tau^k,1]$. Then

$$\frac{E[\bar{R}^k(\tau^k,1)]-E[R^k(\tau^k,1)]}{\sqrt{k}}$$
$$\le G|J|\sum_{j=1}^J r_j E\left[\frac{X_j}{\lambda_j}|\beta_j(1)-\beta_j(\tau^k)| + \frac{X_j}{\lambda_j}\frac{|\epsilon_j^k(1)-\epsilon_j^k(\tau^k)|}{\sqrt{k}} + \left|\phi_j\left(\frac{\Lambda_j^k(1)}{k}\right)-\phi_j\left(\frac{\Lambda_j^k(\tau^k)}{k}\right)\right| + \frac{1}{\sqrt{k}}\left|\delta_j^k\left(\frac{\Lambda_j^k(1)}{k}\right)-\delta_j^k\left(\frac{\Lambda_j^k(\tau^k)}{k}\right)\right|\right].$$

The terms involving $|\beta_j(1)-\beta_j(\tau^k)|$ are covered by (80). The terms involving $|\epsilon_j^k(1)-\epsilon_j^k(\tau^k)|$ are covered by Assumption 3.2. The terms involving $|\delta_j^k(\Lambda_j^k(1)/k)-\delta_j^k(\Lambda_j^k(\tau^k)/k)|$ are covered by Lemma 4.1. The remaining terms, involving $|\phi_j(\Lambda_j^k(1)/k)-\phi_j(\Lambda_j^k(\tau^k)/k)|$, require a bit of effort. By Theorem 4.4, $\tau^k \xrightarrow{P} 1$ as $k\to\infty$ so that

$$\left|\phi_j\left(\frac{\Lambda_j^k(1)}{k}\right)-\phi_j\left(\frac{\Lambda_j^k(\tau^k)}{k}\right)\right| \xrightarrow{P} 0 \quad \text{as } k\to\infty$$

by the a.s. continuity of Brownian motion. To complete the proof, we show uniform integrability of $|\phi_j(\Lambda_j^k(1)/k)-\phi_j(\Lambda_j^k(\tau^k)/k)|$. We can write, using Minkowski's inequality,

$$\sqrt{E\left[\left(\phi_j\left(\frac{\Lambda_j^k(1)}{k}\right)-\phi_j\left(\frac{\Lambda_j^k(\tau^k)}{k}\right)\right)^2\right]}\le\sqrt{E\left[\left(\phi_j\left(\frac{\Lambda_j^k(1)}{k}\right)\right)^2\right]}+\sqrt{E\left[\left(\phi_j\left(\frac{\Lambda_j^k(\tau^k)}{k}\right)\right)^2\right]}.$$

Both terms on the right-hand side are uniformly bounded in k by Lemma 4.1, so uniform integrability follows. Finally, recall that

$$R^k=\sum_{j=1}^J r_j z_j^k(\tau^k)+R^k(\tau^k,1).$$

Combining (83) and (84) yields

$$\frac{E[\bar{R}^k-R^k]}{\sqrt{k}}\to 0 \quad \text{as } k\to\infty. \quad \square$$

If we let $E[R_{\text{opt}}^k]$ denote the expected revenue under the (unknown) optimal policy, we have $E[R^k]\le E[R_{\text{opt}}^k]\le E[\bar{R}^k]$. Thus a consequence of Theorem 4.5 is that the gain of the hindsight optimal policy is $o(\sqrt{k})$:

$$\lim_{k\to\infty}\frac{E[\bar{R}^k]-E[R_{\text{opt}}^k]}{\sqrt{k}}=0.$$

We conclude this section by pointing out that the ideas behind our approach may spawn a family of other admission policies that employ re-solving strategies to achieve asymptotic optimality on the diffusion scale. The critical element of our policy is to choose a right time to reoptimize, which has to be early enough so there remain sufficient capacities and future arrivals to cancel out previous deviations from the hindsight optimum, and late enough so the deviations that occur afterwards become negligible on the diffusion scale. As long as these conditions are met, the reoptimization time does not need to be a random stopping time defined by our triggering mechanism. For instance, one may design a set of open-loop policies by fixing a constant $0<K_0<1$ and for each problem in the sequence $k=1,2,\ldots,$ presetting the re-solving time at

$$t^k=1-\frac{K_0}{k^\alpha},\quad \alpha>0.$$

As a rough estimation, the total deviation from the hindsight optimum accumulated before t^k is on the order of $k^{1/2}$. At t^k, the remaining amounts of capacities and future arrivals are on the order of $k^{1-\alpha}$, so the deviation can be corrected by the reoptimization if $0<\alpha<1/2$. The deviation after t^k is on the order of $k^{(1-\alpha)/2}$. Therefore, by the same argument that supports our policy, one may expect that the above open-loop policies are also optimal on the diffusion scale (with $0<\alpha<1/2$). Nevertheless, the scaling parameter k is only meaningful when we introduce a sequence of problems. So these open-loop policies, which rely on the value of k, are hard to specify and implement for any individual problem instance. This downside needs to be balanced against the upside that there is no need to keep track of the trigger function and thresholds. We leave further investigation of these issues for others to undertake.

5. Numerical studies.

In this section, we discuss the implementation and performance of the T^2 policy through numerical case studies. In §5.1, we numerically demonstrate the asymptotic optimality of the policy and the corresponding revenue improvements. We identify some practical difficulties of applying the policy to small problems in §5.2, and propose solutions in §5.3.

We consider an example with three resources and 11 customer classes. All resources have 100 units of capacity. We assume the arrival process of each customer class is Poisson and generate sample paths of arrivals accordingly. Table 1 gives the arrival rate, price, and resource usage of each class.

5.1. Convergence property and performance advantage of T^2 policy.

We demonstrate the asymptotic optimality of the T^2 policy by comparing it with the LP policy discussed in Cooper [3]. The two policies use the same information about demand (mean arrivals) and employ the same optimization technique (LP). Nevertheless, their performances differ significantly, as shown in the following example.

We scale both customer arrival rates and capacities by a factor of $k=2^K$ ($K=0,1,\ldots,12$). For each k, we generate 100 sample paths, and for each sample path, we simulate customer admission and revenue collection under each policy. The mean revenue under a given policy is estimated by averaging the revenues of all sample

Table 1. Example setup: input table.

			Resource usage		
Class	Arrival rate	Price	1	2	3
1	60	90	1		
2	35	90		1	
3	30	130		1	
4	30	130	1		
5	25	130	1		1
6	21	130		1	1
7	18	50	1		
8	14	90	1		1
9	20	50		1	
10	25	90		1	1
11	16	50			1

paths. The shortfall of the mean revenue from the hindsight optimum is defined as the revenue loss. Figure 1(a) shows that under both policies the revenue loss increases with the problem size.

We divide the revenue loss by the scaling parameter k and plot the normalized values in Figure 1(b). Qualitatively the trends are the same for both policies: as k increases, the normalized revenue loss converges to zero, even though quantitatively the convergence rate is higher under the T^2 policy. The observation is consistent with the result that both T^2 and LP policies are asymptotically optimal on the fluid scale.

We then divide the revenue loss by the square root of the problem size, $\sqrt{k} = 2^{K/2}$. Figure 1(c) shows a qualitative difference between the two policies. As the problem scales, the revenue loss normalized by $\sqrt{k}$ still converges to zero under the T^2 policy, but does not decrease under the LP policy. The figure illustrates our analytical result that the T^2 policy is asymptotically optimal on the diffusion scale, while the LP policy appears not to be.

The advantage of the T^2 policy over the LP policy can be observed even before the problem size is scaled up. Figure 2 shows such a comparison of revenue samples under each policy when the problem is at its original size, i.e., $k = 1$ $(K = 0)$. The revenue collected on each sample path under the T^2 policy is sorted in an ascending

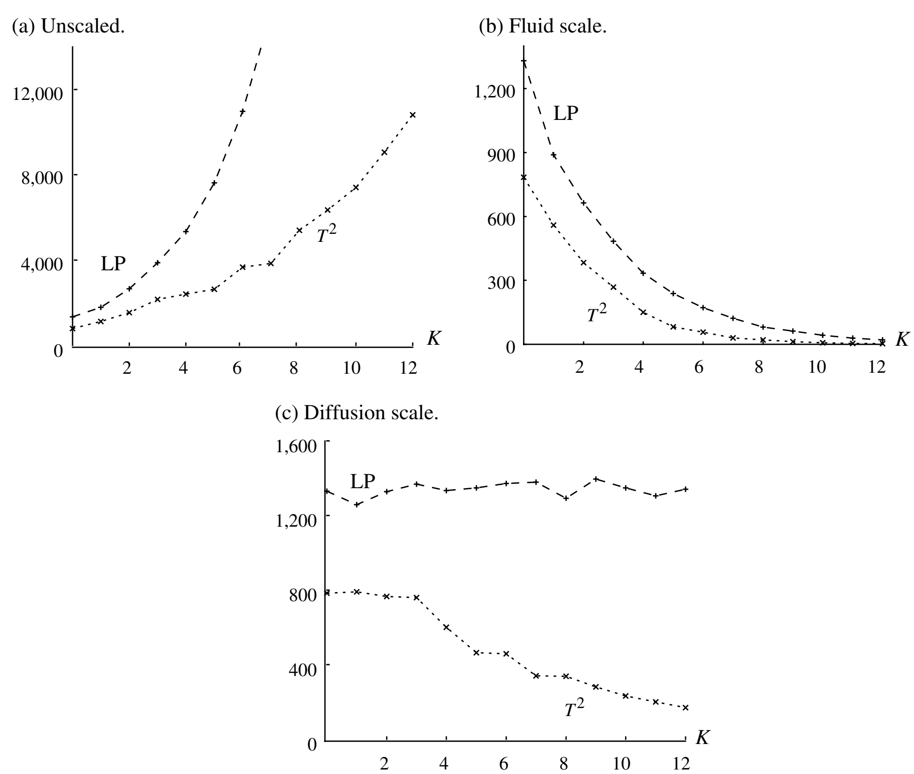

Figure 1. Comparison of revenue loss: LP and T^2.

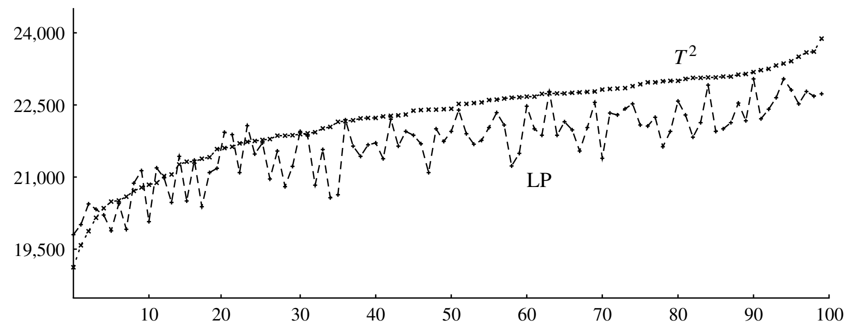

FIGURE 2. Revenue comparison: LP and T^2.

order, and matched with the revenue collected on the same sample path under the LP policy. The figure shows that in most cases the T^2 policy dominates the LP policy. We define the ratio of the revenue obtained under a given policy to the hindsight optimum as the *revenue achievement index*. In this example, the average index over the 100 samples is 96.6% and 94.2% under the T^2 and LP policies, respectively. The t-test shows that this difference is significant at the 0.001 confidence level.

Having demonstrated the advantage of the T^2 policy, we now describe an alternative implementation of its admission rule. Instead of accepting customers according to a given ratio probabilistically, one may take a deterministic approach and associate each customer class with an admission counter. The counter for class j $(j = 1, 2, \ldots, J)$ starts from 0 at $t = 0$, and advances by an increment of X_j/λ_j when a class j customer arrives. The customer is admitted when and only when her class counter is at least one. When a customer is admitted, the counter is decremented by one.

The new approach removes the randomness associated with the probabilistic admission, but introduces rounding errors in the expected values; i.e., the expected number of class j $(j = 1, 2, \ldots, J)$ customers accepted by time t is $E[\lfloor X_j \Lambda_j(t)/\lambda_j \rfloor]$ instead of the intended value of $X_j t$. Nevertheless, as the problem size increases, the impact of the rounding error becomes negligible. In our case studies, when $k = 1$ the deterministic admission rule leads to a slightly worse performance than the probabilistic approach (average revenue achievement indexes are 96.0% and 96.6%, respectively), reflecting the impact of the rounding error. Once $k \geq 8$, the performances of the two approaches are almost identical.

5.2. Limitation of the T^2 policy. Since an asymptotically optimal policy may not be optimal, for a given (finite) system we cannot expect that the T^2 policy can outperform every policy that is not asymptotically optimal on the diffusion scale. While Figure 2 shows that the T^2 policy beats the LP policy when $k = 1$, we provide here an example of a policy that, while not asymptotically optimal on the diffusion scale, has performance close to that of the T^2 policy for small and moderate-sized problems. That policy is the fixed-point (FP) policy of Li and Yao (see Li and Yao [8]). Like the LP policy, the FP policy controls admission based on some preset booking limits that remain unchanged for the whole period, so, by our discussion in §3, we conjecture that it is not asymptotically optimal on the diffusion scale. Nevertheless, the FP policy was developed specifically for Poisson arrivals and uses the distribution function of arrivals (instead of relying only on the mean number of arrivals, as is the case with the LP policy) for optimizing booking limits. As a result, in our example the FP policy gives higher average revenues than the LP policy.

We apply both the T^2 and FP policies to the above problem for $k = 1$, simulate customer admissions on 100 sample paths, and compare revenue samples in Figure 3. Despite the fact that the T^2 policy has better asymptotic behavior than the FP policy, there is no revenue improvement. The t-test also shows that the difference in the revenue achievement index between the two polices is insignificant.

To explain this result, recall from the proofs of Theorems 4.4 and 4.5 that the advantage of the T^2 policy is based on two essential factors: First, reoptimize the admission rule (albeit only once) to correct deviations from the hindsight optimum. Second, to remove all deviations over the entire period on the diffusion scale, the reoptimization time, τ^k, is pushed to the very end of the horizon as $k \to \infty$. If the second condition does not hold and the reoptimization takes place early, then the policy becomes much less effective. For instance, if the reoptimization takes place right at the beginning, then there is hardly any difference between the T^2 policy and a policy that does not reoptimize.

Early reoptimization is likely to happen when the problem size is small. Recall that the reoptimization is triggered when the trigger function $\Gamma^k(t)$ violates a threshold:

$$\text{either} \quad kX_j(1-t)\ \forall\, j\colon X_j > 0 \quad \text{or} \quad k(\lambda_j - X_j)(1-t)\ \forall\, j\colon X_j < \lambda_j.$$

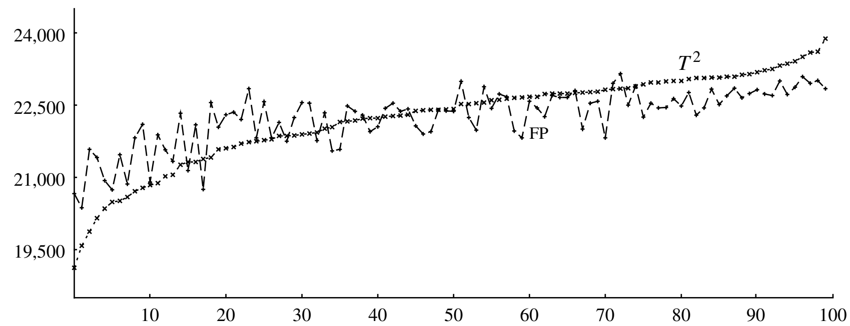

FIGURE 3. Revenue comparison: FP and T^2.

Even though $\Gamma^k(t)$ is on the diffusion scale ($\sqrt{k}$) and all of the thresholds are on the fluid scale (k), the former can still exceed the latter at an early time if k is small. In the above example, the average reoptimization time is 0.12, so the reoptimization does not correct deviations that occurred in the remaining 88% percent of the period (0.12,1]. Therefore, it should not be entirely surprising that the T^2 policy does not significantly outperform the FP policy.

The difficulty disappears as we scale up the problem size. Table 2 gives the average time to reoptimize the admission rule (denoted by $\bar{\tau}^k$; bar stands for average and k indicates the problem size) under the T^2 policy in the above problem when we increase k from 1 to 4,096 ($K = 0, \ldots, 12$). The table shows that $\bar{\tau}^k$ monotonically increases with k and approaches the limit $t = 1$, as indicated by our analytical results in (74).

As the time to reoptimize increases, the advantage of the T^2 policy becomes evident. We perform the same numerical experiments as above for different problem sizes and test the hypothesis that the mean of the revenue achievement index under the T^2 policy is no greater than that under the FP policy. When $k = 1, 2$ ($K = 0, 1$), the p-values of the t-tests are -0.19 and -0.34, respectively, so the hypothesis should clearly be accepted. When $k = 4$ ($K = 2$), the p-value is 1.47 and the hypothesis can be rejected at the 0.1 significance level. When $k = 8$ ($K = 3$), the rejection of the hypothesis is almost certain (at the 0.05 level, the p-value is 1.75), which indicates an observable improvement of the T^2 policy. While we expect the trend will be more obvious as the problem size increases we stopped at this point because the exponential growth of the fixed-point equations makes it difficult to calculate booking limits under the FP policy.

5.3. Strategies for handling small problems. A natural strategy to improve the performance of the T^2 policy for small problems is to allow repeated reoptimization instead of reoptimizing only once, as the T^2 policy does. After each reoptimization, the trigger and threshold functions are reset and restarted for the remaining period. The next reoptimization is invoked whenever the new trigger value exceeds a new threshold. We refer to the new approach as the relaxed T^2 policy, apply it to the same example as in Figure 3, and compare the result with the FP policy in Figure 4(a). The figure shows a visible improvement of revenue under the relaxed T^2 policy.

Once we relax the restriction that the reoptimization can take place only once, another policy that uses an even simpler admission rule, referred to as the open-admission (OA) policy, can be applied. Like the T^2 policy, the OA policy employs a trigger-threshold mechanism (though the trigger and threshold functions are defined differently), and, as the name indicates, accepts all class j ($j = 1, 2, \ldots, J$) customers if $X_j > 0$ (where X_j is an optimal solution to the fluid LP) and rejects all others. This policy leads to excessive acceptance (on the fluid scale) of class j customers if $0 < X_j < \lambda_j$, and relies on reoptimization, which can reset $X_j = 0$ to reject all future arrivals, to eliminate the excess.

To describe the OA policy, let τ_0 be the last time the fluid model was optimized and let $X_j(\tau_0)$ be the solution of the fluid LP. The procedure is as follows:

(i) Initialization: $\tau_0 = 0$ and solve the fluid LP (33).

(ii) Accept all class j customers if $X_j(\tau_0) > 0$. Note that if the fluid LP gives a binary solution, i.e., for each $j = 1, \ldots, J$, either $X_j(\tau_0) = \lambda_j$ or $X_j(\tau_0) = 0$, then the admission rules are equivalent between the T^2 and OA policies.

TABLE 2. Time to reoptimize the admission rule (average of 100 cases).

k	1	2	4	8	16	32	64	128	256	512	1,024	2,048	4,096
$\bar{\tau}^k$	0.12	0.20	0.29	0.41	0.53	0.64	0.73	0.79	0.86	0.89	0.93	0.95	0.96

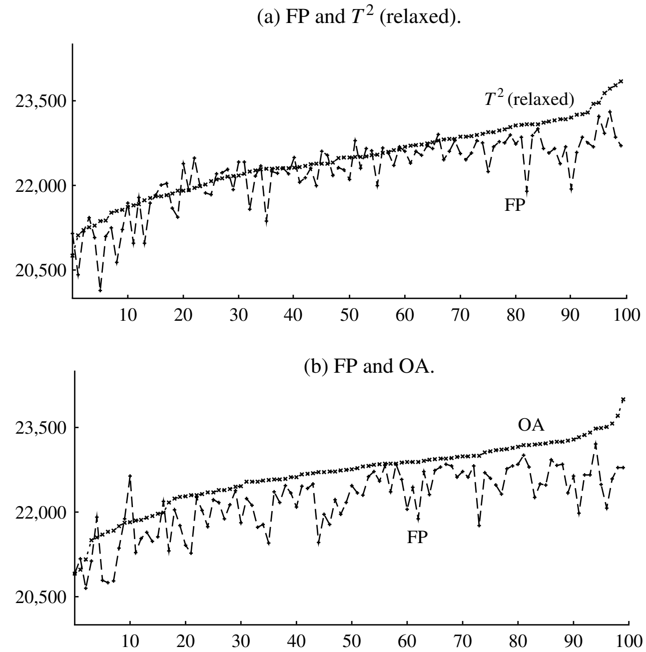

FIGURE 4. Revenue comparison.

(iii) In case of the binary solution, use the same trigger function and thresholds as the T^2 policy. Otherwise, for each resource $l = 1, \ldots, L$, define a trigger function

$$\Gamma_l(t) = \sum_{j=1}^{J} a_{lj}\left[\frac{1-t}{1-\tau_0}X_j(\tau_0) + z_j(t)\right] - C_l, \tag{86}$$

where $z_j(t)$ is the number of class j customers accepted by t, and a threshold

$$s_l(t) = \min_j \left\{ \frac{1-t}{1-\tau_0} a_{lj} X_j(\tau_0) \,\middle|\, a_{lj} X_j(\tau_0) > 0 \right\}. \tag{87}$$

Both trigger and threshold values are updated when an arrival occurs.

(iv) In the case of the nonbinary solution, reoptimize the fluid model whenever $\Gamma_l(t) \geq s_l(t)$ for any l. In the case of the binary solution, the reoptimization is triggered in the same way as under the T^2 policy. In both cases, reset $t = \tau_0$ and go back to Step 2.

For the same example, we plot revenue samples under the OA and FP policies in Figure 4(b), which shows a clear lead of the former. The revenue achievement index is 98.3% and 96.3% for the OA and FP policies, respectively. The difference is at the 0.001 significance level by the t-test.

In Figure 5, we compare the scaled performance of the OA and FP policies to obtain an indication of their asymptotic properties. The layout of the figure is the same as that of Figure 1. Again, we stop at $k = 8$ $(K = 3)$ due to the difficulty of calculating FP booking limits for very large problems. Nevertheless, a clear and consistent trend is observable from this limited number of data points. Under the FP policy, the revenue loss decreases with problem size on the fluid scale but not on the diffusion scale. Under the OA policy, it declines on both scales.

We end our discussion of the OA policy by mentioning an interesting observation. Intuitively, one would expect that, for the same problem, there will be more reoptimizations under the OA policy, which overloads the system, than the relaxed T^2 policy, which does not. To our surprise, we observed the opposite in some examples. The observation can be explained by the difference of trigger functions. Under the OA policy, the trigger function is

$$\Gamma_l(t) = \sum_{j=1}^{J} a_{lj}\left[\frac{1-t}{1-\tau_0}X_j(\tau_0) + z_j(t)\right] - C_l, \quad l = 1, \ldots, L,$$

which allows different customer classes to cancel out each others' random deviations of arrivals. Under the T^2 policy, the trigger function is

$$\Gamma(t) = \alpha \sum_{j \in J_\lambda} |z_j(t) - \lambda_j t|,$$

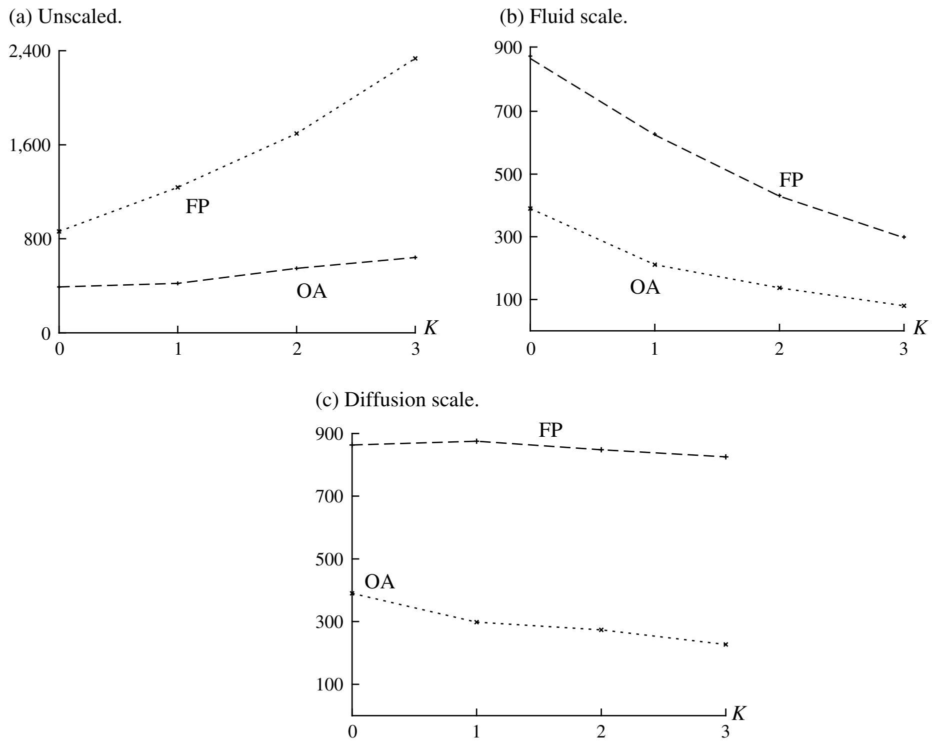

FIGURE 5. Comparison of revenue loss: FP and OA.

which does not have this pooling effect. Even though the deviations are on the diffusion scale, for small problems their cancellations can still be a significant factor compared with the fluid-scale bias associated with the OA policy, which makes the observed effect possible.

Appendix.

PROOF OF LEMMA 3.1. Due to the independence of the J renewal processes, it is sufficient to prove this result for each j separately. To simplify the notation we drop the subscript j.

Combining Equations (6) and (10), we can write

$$\beta^k(t) = \beta(t) + k^{-1/2}\epsilon^k(t), \quad k \geq 1, \; 0 \leq t \leq 1.$$

Thus, using Minkowski's inequality (cf. Durrett [4]) with $p = 2$, we have

$$\begin{aligned} \left\{E\left[\left(\sup_{0\le t\le 1}|k^{-1/2}\epsilon^k(t)|\right)^2\right]\right\}^{1/2} &= \left\{E\left[\left(\sup_{0\le t\le 1}|\beta^k(t)-\beta(t)|\right)^2\right]\right\}^{1/2} \\ &\le \left\{E\left[\left(\sup_{0\le t\le 1}|\beta^k(t)|\right)^2\right]\right\}^{1/2} + \left\{E\left[\left(\sup_{0\le t\le 1}|\beta(t)|\right)^2\right]\right\}^{1/2}. \end{aligned} \tag{88}$$

Let $\mathscr{C}_1 \equiv E[(\sup_{0\le t\le 1}|\beta(t)|)^2]$. Then $\mathscr{C}_1 < \infty$. So we focus on the first term on the right-hand side of (88). We want to show that

$$\sup_{k\ge 1} E\left[\left(\sup_{0\le t\le 1}|\beta^k(t)|\right)^2\right] < \infty.$$

We prove this as follows: By Equations (6) and (7), we have

$$\beta^k(t) = \frac{\Lambda(kt) - kt\lambda}{\sqrt{k}}.$$

Let ξ_i, $i \geq 1$ denote the interarrival times of the renewal process $\Lambda = \{\Lambda(t),\, t \geq 0\}$. We can write

$$\Lambda(kt) + 1 - kt\lambda = \sum_{i=1}^{\Lambda(kt)+1}(1-\lambda\xi_i) + \lambda\left(\sum_{i=1}^{\Lambda(kt)+1}\xi_i - kt\right),$$

so that

$$E\left[\left(\sup_{0\le t\le 1}|\beta^k(t)|\right)^2\right]\le 4E\left[\sup_{0\le t\le 1}\left(k^{-1/2}\sum_{i=1}^{\Lambda(kt)+1}(1-\lambda\xi_i)\right)^2\right]+2\frac{\lambda^2}{k}E\left[\sup_{0\le t\le 1}\left(\sum_{i=1}^{\Lambda(kt)+1}\xi_i-kt\right)^2\right]+\frac{4}{k}.$$

(For any real A, B, C, $(A+B+C)^2\le 4A^2+4B^2+2C^2$.)

Let $\mathsf{F}=(\mathscr{F}(t),\ t\ge 0)$ be a filtration such that ξ_n is $\mathscr{F}(n)$ measurable and ξ_n is independent of $\mathscr{F}(n-1)$, $n\ge 1$. By Lemma 2 of Coffman et al. [2], for every $t\ge 0$, $\Lambda(kt)+1$ is a stopping time with respect to F. Let $\widehat{\mathsf{F}}_k=(\widehat{\mathscr{F}}_k(t),\ t\ge 0)$ with $\widehat{\mathscr{F}}_k(t)=\mathscr{F}(\Lambda(kt)+1)$. For $n\ge 1$, let $\tau_n^k=\inf\{t\ge 0\colon \Lambda(kt)\ge kn\}$. Then $\{\tau_n^k,\ n\ge 1\}$ are stopping times for $\widehat{\mathsf{F}}_k$. Let

$$m_n^k(t)=\sum_{i=1}^{\Lambda(kt\wedge\tau_n^k)+1}(1-\lambda\xi_i)$$

for $n\ge 1$, $k\ge 1$ and $0\le t\le 1$, and let $m_n^k=(m_n^k(t),\ t\ge 0)$. Then, by the proof of Lemma 2 of Coffman et al. [2], m_n^k is a square integrable martingale. Thus, by Doob's inequality (c.f. Ethier and Kurtz [5], Proposition 2.16(b)),

$$E\left[\sup_{0\le t\le 1}(m_n^k(t))^2\right]\le 4E[(m_n^k(1))^2]. \tag{89}$$

The predictable quadratic variation process of m_n^k, $\langle m_n^k\rangle=(\langle m_n^k\rangle(t),\ t\ge 0)$, is given by

$$\langle m_n^k\rangle(t)=\lambda^2\sigma^2\Lambda(kt\wedge\tau_n^k), \tag{90}$$

where $\sigma^2=\mathrm{Var}(\xi_1)$. Let $L_n^k(t)=(m_n^k(t))^2-\langle m_n^k\rangle(t)$. Then, by the definition of predictable quadratic variation, $(L_n^k(t),\ t\ge 0)$ is a martingale with $L_n^k(0)=(1-\lambda\xi_1)^2$ so that $E[L_n^k(0)]=\lambda^2\sigma^2$. Thus

$$E[(m_n^k(1))^2]=\lambda^2\sigma^2(E[\Lambda(k\wedge\tau_n^k)]+1)\le\lambda^2\sigma^2(E[\Lambda(k)]+1). \tag{91}$$

Combining (89) and (91) yields $E[\sup_{0\le t\le 1}(m_n^k(t))^2]\le 4\lambda^2\sigma^2(E[\Lambda(k)]+1)$. By standard results on renewal processes, $\tau_n^k\to\infty$ a.s. as $n\to\infty$, so by Fatou's Lemma,

$$\begin{aligned}E\left[\sup_{0\le t\le 1}\left(k^{-1/2}\sum_{i=1}^{\Lambda(kt)+1}(1-\lambda\xi_i)\right)^2\right]&\le k^{-1}\liminf_{n\to\infty}E\left[\sup_{0\le t\le 1}(m_n^k(t))^2\right]\\&\le 4\lambda^2\sigma^2\frac{E[\Lambda(k)]+1}{k}.\end{aligned}$$

By the renewal theorem, $k^{-1}(E[\Lambda(k)]+1)\to\lambda$ as $k\to\infty$, so

$$\sup_{k\ge 1}E\left[\sup_{0\le t\le 1}\left(k^{-1/2}\sum_{i=1}^{\Lambda(kt)+1}(1-\lambda\xi_i)\right)^2\right]<\infty.$$

We now deal with the term $E[\sup_{0\le t\le 1}(\sum_{i=1}^{\Lambda(kt)+1}\xi_i-kt)^2]$. We can write $\sum_{i=1}^{\Lambda(kt)+1}\xi_i-kt\le\xi_{\Lambda(kt)+1}$ so that

$$\sup_{0\le t\le 1}\left(\sum_{i=1}^{\Lambda(kt)+1}\xi_i-kt\right)^2\le\sup_{1\le i\le\Lambda(k)+1}\xi_i^2.$$

We have the simple bound $E[\sup_{1\le i\le\Lambda(k)+1}\xi_i^2]\le E\sum_{i=1}^{\Lambda(kt)+1}\xi_i^2$. By Wald's Lemma ($\Lambda(kt)+1$ is a stopping time with respect to F and $E[\Lambda(kt)+1]$ is finite for $kt<\infty$), $E[\sum_{i=1}^{\Lambda(kt)+1}\xi_i^2]=E[\Lambda(kt)+1]E[\xi_1^2]$. Thus

$$E\left[\sup_{0\le t\le 1}\left(\sum_{i=1}^{\Lambda(kt)+1}\xi_i-kt\right)^2\right]\le E[\Lambda(kt)+1]E[\xi_1^2],$$

and $\sup_{k\ge 1}k^{-1}E[\sup_{0\le t\le 1}(\sum_{i=1}^{\Lambda(kt)+1}\xi_i-kt)^2]<\infty$.

By (88) we thus know that $\sup_{k\ge 1}E[(\sup_{0\le t\le 1}|k^{-1/2}\epsilon^k(t)|)^2]<\infty$, which implies that $\{\sup_{0\le t\le 1}|k^{-1/2}\epsilon^k(t)|,\ k\ge 1\}$ is uniformly integrable. Taken together with (11), this proves the lemma. □

Proof of Equation (32). Let $X_1^{k,p} = kC$ and let $X_2^{k,p} = k(C-\lambda_1)$. Note that $X_2^{k,p} > 0$ since $\lambda_1 < C$.

Let $z_j^{k,p}$ and $\bar{z}_j^k$ denote the total number of class j customers accepted by policy p and the hindsight policy, respectively. Then

$$\begin{aligned} \bar{z}_1^k &= \Lambda_1^k(1) \wedge kC, \\ z_1^{k,p} &\geq \Lambda_1^k(1) \wedge (kC - X_2^{k,p}) = \Lambda_1^k(1) \wedge k\lambda_1. \end{aligned} \tag{92}$$

Since $\lambda_1 + \lambda_2 > C$, a.s. for k large enough,

$$\begin{aligned} \bar{z}_2^k &= \Lambda_2^k(1) \wedge (kC - \bar{z}_1^k) = kC - \bar{z}_1^k, \\ z_2^{k,p} &= \Lambda_2^k(1) \wedge X_2^{k,p} \wedge (kC - z_1^{k,p}) = (kC - k\lambda_1) \wedge (kC - z_1^{k,p}) \\ &= (kC - z_1^{k,p}) - (k\lambda_1 - z_1^{k,p})^+ \\ &\geq (kC - z_1^{k,p}) - (k\lambda_1 - \Lambda_1^k(1))^+. \end{aligned} \tag{93}$$

As a result,

$$\begin{aligned} \bar{R}^k &= r_2 kC + (r_1 - r_2)\bar{z}_1^k, \\ R^{k,p} &\geq r_2 kC + (r_1 - r_2) z_1^{k,p} - r_2(k\lambda_1 - \Lambda_1^k(1))^+. \end{aligned} \tag{94}$$

Therefore,

$$\begin{aligned} \lim_{k\to\infty} \frac{\bar{R}^k - R^{k,p}}{\sqrt{k}} &\leq \lim_{k\to\infty} \frac{(r_1 - r_2)|\bar{z}_1^k - z_1^{k,p}| + r_2[k\lambda_1 - \Lambda_1^k(1)]^+}{\sqrt{k}} \\ &\leq (r_1 + r_2) \lim_{k\to\infty} \frac{|k\lambda_1 - \Lambda_1^k(1)|}{\sqrt{k}} = 0. \end{aligned} \tag{95}$$

With renewal arrivals, $\sigma_1^2 = 0$ implies that the interarrival times are deterministic so that $|k\lambda_1 - \Lambda_1^k(1)| \leq 1$ for all $k > 0$, which implies that $E|k\lambda_1 - \Lambda_1^k(1)| \leq 1$ and

$$\lim_{k\to\infty} \frac{E[\bar{R}^k] - E[R^{k,p}]}{\sqrt{k}} \leq (r_1 + r_2) \lim_{k\to\infty} \frac{E|k\lambda_1 - \Lambda_1^k(1)|}{\sqrt{k}} = 0. \quad \square$$

Proof of Lemma 4.1. (a) We can write

$$E\left[\left(\phi_j\left(\frac{\Lambda_j^k(t)}{k}\right)\right)^2\right] = E\left\{E\left[\left(\phi_j\left(\frac{\Lambda_j^k(t)}{k}\right)\right)^2 \middle| \Lambda_j^k(t)\right]\right\} = \sigma_j^2 E\left[\frac{\Lambda_j^k(t)}{k}\right].$$

Note that $\beta_j(t)$ in (10) is a driftless Brownian Motion, so that

$$E\left[\frac{\Lambda_j^k(t)}{k}\right] = t\lambda_j + \frac{E(\epsilon_j^k(t))}{k}.$$

The first term on the right-hand side does not depend on k, and the second term converges to 0 as $k \to \infty$. So (a) follows.

(b) This follows by a slight variation of the proof of (a). We can write $E[(\phi_j(\Lambda_j^k(\tau^k)/k))^2] = E\{E[(\phi_j(\Lambda_j^k(\tau^k)/k))^2 | \Lambda_j^k(\tau^k)]\} = \sigma_j^2 E[\Lambda_j^k(\tau^k)/k]$. By (10),

$$E\left[\frac{\Lambda_j^k(\tau^k)}{k}\right] = \lambda_j E[\tau^k] + \frac{E(\beta_j(\tau^k))}{\sqrt{k}} + \frac{E(\epsilon_j^k(\tau^k))}{k}.$$

Since $\tau^k \leq 1$ and the upper bound of the second term ($\sup_{0\leq t\leq 1} \beta_j(t)/\sqrt{k}$) is decreasing in k, the above is again uniformly bounded in k.

(c) For convenience, we drop the subscript j in this part of the proof. Take $0 \leq t \leq 1$. We have $\phi^k(\Lambda^k(t)/k) = \phi(\Lambda^k(t)/k) + \delta^k(\Lambda^k(t)/k)/\sqrt{k}$, so that, by Minkowski's inequality,

$$\begin{aligned} \left[E\left(\left|\frac{\delta^k(\frac{\Lambda^k(t)}{k})}{\sqrt{k}}\right|^{3/2}\right)\right]^{2/3} &= \left[E\left(\left|\phi^k\left(\frac{\Lambda^k(t)}{k}\right) - \phi\left(\frac{\Lambda^k(t)}{k}\right)\right|^{3/2}\right)\right]^{2/3} \\ &\leq \left[E\left(\left|\phi^k\left(\frac{\Lambda^k(t)}{k}\right)\right|^{3/2}\right)\right]^{2/3} + \left[E\left(\left|\phi\left(\frac{\Lambda^k(t)}{k}\right)\right|^{3/2}\right)\right]^{2/3}. \end{aligned} \tag{96}$$

By part (a) of this lemma, the second term in the last line is bounded uniformly in k. Let $\mathscr{C}_\phi$ denote a finite bound. By Chebyshev's inequality, for $M > 0$,

$$P\left(\left|\phi^k\left(\frac{\Lambda^k(t)}{k}\right)\right| \geq M\right) \leq \frac{1}{M^2}E\left|\phi^k\left(\frac{\Lambda^k(t)}{k}\right)\right|^2$$
$$= \frac{1}{M^2}E\left\{E\left[\left(\phi^k\left(\frac{\Lambda^k(t)}{k}\right)\right)^2 \middle| \Lambda^k(t)\right]\right\} \leq \frac{\sigma^2}{M^2}E\left[\frac{\Lambda^k(t)}{k}\right] \leq \frac{\sigma^2}{M^2}\mathscr{C}_\Lambda, \tag{97}$$

where $\mathscr{C}_\Lambda \equiv \lambda + E[\sup_{0\leq t\leq 1}|\beta(t)|]/\sqrt{k} + \sup_{k\geq 1}\sup_{0\leq t\leq 1} E|\epsilon^k(t)|/k < \infty$. Thus

$$E\left(\left|\phi^k\left(\frac{\Lambda^k(t)}{k}\right)\right|^{3/2}\right) = \int_0^\infty P\left(\left|\phi^k\left(\frac{\Lambda^k(t)}{k}\right)\right|^{3/2} \geq x\right)dx$$
$$\leq 1 + \int_1^\infty P\left(\left|\phi^k\left(\frac{\Lambda^k(t)}{k}\right)\right| \geq x^{2/3}\right)dx$$
$$\leq 1 + \mathscr{C}_\Lambda\sigma^2\int_1^\infty x^{-4/3}\,dx = 1 + 3\mathscr{C}_\Lambda\sigma^2. \tag{98}$$

So now we can write $[E(|\delta^k(\Lambda^k(t)/k)/\sqrt{k}|^{3/2})]^{2/3} \leq \mathscr{C}_\phi + [1+3\mathscr{C}_\Lambda\sigma^2]^{2/3}$, which proves (c).

(d) This proof is similar to that of part (c). Analogously to (96), we have

$$\left[E\left(\left|\frac{\delta^k(\frac{\Lambda^k(\tau^k)}{k})}{\sqrt{k}}\right|^{3/2}\right)\right]^{2/3} = \left[E\left(\left|\phi^k\left(\frac{\Lambda^k(\tau^k)}{k}\right) - \phi\left(\frac{\Lambda^k(\tau^k)}{k}\right)\right|^{3/2}\right)\right]^{2/3}$$
$$\leq \left[E\left(\left|\phi^k\left(\frac{\Lambda^k(\tau^k)}{k}\right)\right|^{3/2}\right)\right]^{2/3} + \left[E\left(\left|\phi\left(\frac{\Lambda^k(\tau^k)}{k}\right)\right|^{3/2}\right)\right]^{2/3}. \tag{99}$$

By part (b) of this lemma, the second term in the last line is bounded uniformly in k. Let $\mathscr{C}'_\phi$ denote a finite bound. Analogously to (97), we have

$$P\left(\left|\phi^k\left(\frac{\Lambda^k(\tau^k)}{k}\right)\right| \geq M\right) \leq \frac{1}{M^2}\sigma^2\mathscr{C}_\Lambda, \tag{100}$$

so that

$$E\left(\left|\phi^k\left(\frac{\Lambda^k(\tau^k)}{k}\right)\right|^{3/2}\right) \leq 1 + 3\mathscr{C}_\Lambda\sigma^2, \tag{101}$$

which allows us to write

$$\left[E\left(\left|\frac{\delta^k(\frac{\Lambda^k(\tau^k)}{k})}{\sqrt{k}}\right|^{3/2}\right)\right]^{2/3} \leq \mathscr{C}'_\phi + [1+3\mathscr{C}_\Lambda\sigma^2]^{2/3},$$

proving (d). □

Proof of Theorem 4.1. Let $\hat{z}_j^k$ be the total number of thinned arrivals of class j customers; i.e.,

$$\hat{z}_j^k = kX_j + \sqrt{k}\frac{X_j}{\lambda_j}\beta_j(1) + \frac{X_j}{\lambda_j}\epsilon_j^k(1) + \sqrt{k}\phi_j\left(\frac{\Lambda_j^k(1)}{k}\right) + \delta_j^k\left(\frac{\Lambda_j^k(1)}{k}\right). \tag{102}$$

For each $j = 1, 2, \ldots, J$, let c_{lj}^k (l: $a_{lj} > 0$) be the minimum possible amount of resource l available to serve class j customers under the probabilistic acceptance rule. Then

$$c_{lj}^k \geq kC_l - \sum_{j'\neq j} a_{lj'}\hat{z}_{j'}^k$$
$$= kC_l - \sum_{j'\neq j} a_{lj'}\left[kX_{j'} + \sqrt{k}\frac{X_{j'}}{\lambda_{j'}}\beta_{j'}(1) + \frac{X_{j'}}{\lambda_{j'}}\epsilon_{j'}^k(1) + \sqrt{k}\phi_{j'}\left(\frac{\Lambda_{j'}^k(1)}{k}\right) + \delta_{j'}^k\left(\frac{\Lambda_{j'}^k(1)}{k}\right)\right]$$
$$\geq a_{lj}kX_j - \sum_{j'\neq j} a_{lj'}\left[\sqrt{k}\frac{X_{j'}}{\lambda_{j'}}\beta_{j'}(1) + \frac{X_{j'}}{\lambda_{j'}}\epsilon_{j'}^k(1) + \sqrt{k}\phi_{j'}\left(\frac{\Lambda_{j'}^k(1)}{k}\right) + \delta_{j'}^k\left(\frac{\Lambda_{j'}^k(1)}{k}\right)\right]. \tag{103}$$

Let z_j^k be the number of class j customers actually accepted under the probabilistic acceptance rule. Then

$$\begin{aligned} z_j^k &\geq \Big(\min_{l:\, a_{lj}>0} c_{lj}^k/a_{lj}\Big) \wedge \hat{z}_j^k \\ &\geq kX_j - G\sum_{j=1}^{J}\bigg[\sqrt{k}\frac{X_{j'}}{\lambda_{j'}}|\beta_{j'}(1)| + \frac{X_{j'}}{\lambda_{j'}}|\epsilon_{j'}^k(1)| + \sqrt{k}\bigg|\phi_{j'}\bigg(\frac{\Lambda_{j'}^k(1)}{k}\bigg)\bigg| + \bigg|\delta_{j'}^k\bigg(\frac{\Lambda_{j'}^k(1)}{k}\bigg)\bigg|\bigg], \end{aligned} \tag{104}$$

where $G = \max_{l,\, j}\{a_{lj}\}/\min_{a_{lj}>0}\{a_{lj}\}$ is a constant independent of k.

Note that $E[R^k] = \sum_{j=1}^{J} r_j E[z_j^k]$ is the expected revenue under the probabilistic acceptance rule. Because $k^{-1}(E[z_1^k], \ldots, E[z_J^k])$ is a feasible solution of the fluid LP, $E[\bar{R}^k] \leq k\sum_{j=1}^{J} r_j X_j$, so (104) implies that

$$E[\bar{R}^k] - E[R^k] \leq G|J|\sum_{j=1}^{J} r_j E\bigg[\sqrt{k}\frac{X_j}{\lambda_j}|\beta_j(1)| + \frac{X_j}{\lambda_j}|\epsilon_j^k(1)| + \sqrt{k}\bigg|\phi_j\bigg(\frac{\Lambda_j^k(1)}{k}\bigg)\bigg| + \bigg|\delta_j^k\bigg(\frac{\Lambda_j^k(1)}{k}\bigg)\bigg|\bigg]. \tag{105}$$

For $j = 1, 2, \ldots, J$, $E(|\beta_j(1)|)$ is finite so that $E(|\beta_j(1)|)/\sqrt{k} \to 0$ as $k \to \infty$, $E(|\epsilon_j^k(1)|)/k \to 0$ as $k \to \infty$ by Assumption 3.2, $E(|\phi_j(\Lambda_j^k(1)/k)|)/\sqrt{k} \to 0$ as $k \to \infty$ by Lemma 4.1, and $E(|\delta_j^k(\Lambda_j^k(1)/k)|)/k \to 0$ as $k \to \infty$ by Lemma 4.1. Thus

$$\lim_{k\to\infty}\frac{E[\bar{R}^k] - E[R^k]}{k} = 0. \quad \square \tag{106}$$

Acknowledgments. The authors are grateful to Tolya Puhalskii for his help on the proof of Lemma 3.1. This enabled the authors to improve on the previous version of the lemma, where a finite fourth moment of interarrival times was assumed. The authors would also like to thank Jim Dai, Anton Kleywegt, and Yongbo Xiao for their helpful comments.

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