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On the use of independent base-stock policies in assemble-to-order inventory systems with nonidentical lead times

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ABSTRACT

We consider the use of Independent Base Stock (IBS) replenishment policies in Assemble-to-Order (ATO) inventory systems. These policies are appealingly simple and widely used, but generally suboptimal for systems with non-identical lead times. We present an IBS policy and prove that its loss of optimality is limited by the ratio of the longest lead time to the shortest one. Our results suggest that IBS policies can work well for systems where differences between lead times are dominated by their lengths.

1. Introduction

Assemble-to-Order (ATO) manufacturing keeps inventory at the component level and assembles a final product only after the demand for it has arrived. Managing an ATO inventory system involves two control policies. The replenishment policy determines the ordering of components. The allocation policy distributes components to serve different demands. Minimizing the long-run average expected total inventory cost is a common objective that we consider in this paper.

Independent base-stock (IBS) policies are probably the most popular replenishment control schemes (e.g., see [9] for a review of the related literature). An IBS policy keeps each component's inventory position, defined as the difference between its on-hand plus on-order inventory and its backlog, at a constant level for all time. In systems with identical replenishment lead times, IBS policies can be asymptotically optimal in the long-lead time or high-volume asymptotic regimes [6]. They also perform well outside these regimes and are even optimal in some special cases [2].

IBS policies become inadequate in systems with non-identical lead times, even for those that contain only one product [7,11]. Nevertheless, finding a better alternative is also hard. Policies that vary inventory positions of some components to satisfy certain optimality criteria have been considered [3,5,7], but their developments have been so far limited to systems with special Bill of Materials (BOM). As a result, base stock policies, such as the ones that Ignore Simultaneous Stock-outs (ISS) of components [10], are still commonly employed in practice. This paper considers a family of IBS policies with different choices of base stock levels from existing schemes such as ISS. We show that *in many ATO systems with a general BOM and deterministic but different lead times, these policies keep the long-run average inventory cost close to its minimum*.

Previous studies have used stochastic programs (SP) to set a lower bound on the average inventory cost of ATO systems [5,6,10]. We start from a similar launching pad by formulating an SP and proving its optimal solution is below the inventory cost under any feasible policy. However, unlike the bound in [5], ours is based on a two-stage SP instead of a $K+1$ stage SP (where K is the number of different lead times). Unlike the bound in [6], ours applies to systems with any number of distinct lead times rather than one lead time only. Unlike the bound in [10], ours covers all feasible policies, not just IBS policies. Inventory control is optimal if it keeps the total expected cost at the lower bound for all time. This happens if, in the ATO system, the replenishment policy replicates the probability distribution of 'component balance' (which we define at the beginning of Section 4.2) of the first stage of the SP; and for any given realization of component balance, the allocation policy replicates the outcome of the second stage of the SP (a condition we refer to as *perfect allocation*).

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A perfect match of component balance between an ATO system *with nonidentical lead times* and its corresponding SP is generally impossible. Our key result, Theorem 2 presents an upper bound on the cost impact of the mismatch for systems under control of our IBS policy. The bound provides a useful performance assessment of the cost of following IBS policies in general. Moreover, for systems where variations of component lead times are dominated by their lengths, the bound implies that our policy entails little loss of optimality. Hence, the cost objective stays close to the achievable minimum under perfect allocation, which is attainable in some special cases [2,4,5].

While perfect allocation is unattainable in general ATO systems, a family of asymptotically-optimal allocation policies has been developed in [6]. These policies involve solving a certain linear program, whose parameters depend on the current state of the system (Section 5 provides more details). For systems with identical lead time, the percentage difference of the long-run average inventory cost between these policies and the perfect allocation converges to zero as the lead time grows [6]. We consider a joint use of these allocation policies and our IBS replenishment policy in systems with nonidentical lead times. We prove that the combination is asymptotically optimal as the lead times grow while their differences grow at a slower rate.

The rest of the paper is divided into four sections. We define the problem in Section 2, present our IBS policy in Section 3, derive the lower bound and carry out performance analysis in Section 4, and conclude the paper with a discussion on the implications of our results in Section 5.

2. Problem formulation

We develop our analysis for the continuous-review formulation. Our results also extend to the periodic-review model by analogous arguments.

We consider ATO systems with m products, n components, and a non-negative integer matrix A as its BOM. Here a_{ji} denotes the amount of component j $(1 \le j \le n)$ used by product i $(1 \le i \le m)$, so row j of A, denoted by $\mathbf{A}_j$, represents the usage of components j $(1 \le j \le n)$ by all products. There are K distinct lead times with $L_K > L_{K-1} \cdots > L_1 > 0$, and we use k_j $(k_j = 1, \dots, K)$ to refer to the index of the lead time of component j. Let $n_0 = 0$ and n_k be the number of components with lead time L_k or shorter $(1 \le k \le K)$, so $n_K = n$. Let components be indexed according to the ascending order of their lead times, so $\{n_{k-1}+1, \dots, n_k\}$ is the index set of components with lead time $L_k (1 \le k \le K)$. The usage of components with lead time L_k is given by A^k, which is a submatrix of A that contains rows $j = n_{k-1}+1, \dots, n_k$ $(k = 1, \dots, K)$ and columns $i = 1, \dots, m$.

Demand arrives according to a compound Poisson process. The number of orders during the time interval $[0, t]$ is denoted by $\Lambda(t)$ $(t \ge 0)$ and $\underset{\sim}{\lambda} = \mathbf{E}[\Lambda(1)]$ is the order arrival rate. There is an associated i.i.d. sequence of random vectors that provide order sizes. A generic element of this sequence is denoted by $\mathbf{S} = (S_1, S_2, \dots, S_m)$, where S_i is the order size of product i, $1 \le i \le m$. We assume that $\mathbf{S}$ has a finite second moment. (In Section 5, for one of our results we assume that $\mathbf{S}$ has a finite moment of order $2+\delta$ where δ is a positive value that can be arbitrarily small.) The total demand during $[0, t]$ is denoted by

$$\mathcal{D}(t) = (\mathcal{D}_1(t), \dots, \mathcal{D}_m(t))', \quad t \ge 0,$$

with $\mathbf{E}[\mathcal{D}(1)] = \boldsymbol{\mu} < \infty$. Define

$$\mathbf{D}(t_1, t_2) = \mathcal{D}(t_2) - \mathcal{D}(t_1), \quad 0 \le t_1 \le t_2,$$

as the demand in the interval $(t_1, t_2]$. For $t \ge L_K$, let

$$\mathbf{D}^k(t) = \mathbf{D}(t - L_k, t - L_{k-1}), \quad 1 \le k \le K, \quad \text{and}$$
$$\bar{\mathbf{D}}^k(t) = \mathbf{D}(t - L_K, t - L_k), \quad 0 \le k \le K,$$

where $L_0 = 0$ (note that $\bar{\mathbf{D}}^0(t) = \mathbf{D}(t - L_K, t)$). Since the demand process is stationary, we can define random vectors $\mathbf{D}^k$ and $\bar{\mathbf{D}}^k$ such that

$$\mathbf{D}^k \stackrel{d}{=} \mathbf{D}^k(t) \quad (1 \le k \le K) \quad \text{and}$$
$$\bar{\mathbf{D}}^k \stackrel{d}{=} \bar{\mathbf{D}}^k(t) \quad (0 \le k \le K), t \ge L_K.$$

Let $\mathcal{R}_j(t)$ be the quantity of component j ordered from the supplier between time $-L_{k_j}$ and time t for $t \ge -L_{k_j}$ and define

$$\mathcal{R}(t) = (\mathcal{R}_1(t), \dots, \mathcal{R}_n(t))', \quad t \ge 0.$$

Denote the total quantity of demand served during $[0, t]$ by

$$\mathcal{Z}(t) = (\mathcal{Z}_1(t), \dots, \mathcal{Z}_m(t))', \quad t \ge 0.$$

Similarly, for $1 \le k \le K$, denote

$$\mathbf{R}^k(t) = \mathcal{R}(t) - \mathcal{R}(t - L_k) \quad \text{and} \quad \mathbf{Z}^k(t) = \mathcal{Z}(t) - \mathcal{Z}(t - L_k), \quad t \ge 0.$$

We consider a backlog model and denote the backlog levels at time t by $\mathbf{B}(t) = (B_1(t), \dots, B_m(t))'$, and the per-unit backlog costs by $\mathbf{b} = (b_1, \dots, b_m)'$. Denote the inventory levels of components with lead time L_k by $\mathbf{I}^k(t) = (I_{n_{k-1}+1}(t), \dots, I_{n_k}(t))'$, and the corresponding unit inventory holding costs by $\mathbf{h}^k = (h_{n_{k-1}+1}, \dots h_{n_k})'$ $(1 \le k \le K)$. Let

$$\mathbf{I}(t) = (I_1(t), \dots, I_n(t))' \quad (t \ge 0) \quad \text{and} \quad \mathbf{h} = (h_1, \dots, h_n)'$$

be concatenations of vectors $\mathbf{I}^k(t)$ $(1 \le k \le K)$ and $\mathbf{h}^k$ $(1 \le k \le K)$ respectively. Changes of backlog and inventory levels are governed by

$$\begin{aligned} &\mathbf{B}(t) = \mathbf{B}(t - L_k) + \mathbf{D}(t - L_k, t) - \mathbf{Z}^k(t), \quad 1 \le k \le K,\ t \ge L_K, \\ \text{and} \quad &\mathbf{I}^k(t) = \mathbf{I}^k(t - L_k) + \mathbf{R}^k(t - L_k) - A^k\mathbf{Z}^k(t), \quad 1 \le k \le K,\ t \ge L_K. \end{aligned} \tag{1}$$

The objective is to minimize the long-run average expected inventory cost

$$\mathcal{C} \equiv \limsup_{T \to \infty} \frac{1}{T} \int_{L_K}^{T+L_K} \mathcal{C}(t)dt, \tag{2}$$

where $\mathcal{C}(t) = \mathbf{b} \cdot \mathbf{E}[\mathbf{B}(t)] + \mathbf{h} \cdot \mathbf{E}[\mathbf{I}(t)] = \mathbf{b} \cdot \mathbf{E}[\mathbf{B}(t)] + \sum_{k=1}^{K} \mathbf{h}^k \cdot \mathbf{E}[\mathbf{I}^k(t)]$.

In our discussion below, for any positive integer l, $\mathbb{R}_l$ and $\mathbb{R}_l^+$ respectively denote the sets of l-dimensional real vectors and nonnegative real vectors.

3. Policy development

We define an IBS replenishment policy with the following base stock levels

$$\mathbf{Y}^k = \mathbf{y}^{k*} + (L_k - L_1)A^k\boldsymbol{\mu}, \quad 1 \le k \le K. \tag{3}$$

Here $\mathbf{y}^{k*} = (y^*_{n_{k-1}+1}, \dots, y^*_{n_k})'$ $(1 \le k \le K)$ are subvectors of $\mathbf{y}^* = (y^*_1, \dots, y^*_n)'$, the optimal solution of the following two-stage stochastic program (SP)

$$\tilde{\Phi} = \min_{\mathbf{y} \in \mathbb{R}_n}\{\mathbf{h} \cdot \mathbf{y} + \mathbf{E}[\tilde{\Phi}^0(\mathbf{y}, \mathbf{D}^1)]\} \tag{4}$$

$$\text{where } \tilde{\Phi}^0(\mathbf{y}, \mathbf{x}) = -\max_{\mathbf{z} \in \mathbb{R}_m}\{\mathbf{c} \cdot \mathbf{z} | \mathbf{z} \le \mathbf{x}, A\mathbf{z} \le \mathbf{y}\}, \tag{5}$$

and $\mathbf{c} = \mathbf{b} + A^T\mathbf{h}$. Our policy is a generalization of an IBS policy defined in [6]. In [6], all components have the same lead time L_1, and $\mathbf{y}^{k*}$ $(1 \le k \le K)$ are prescribed as base stock levels to serve demands occurring over a period of that length. These base stock levels are inadequate for our case, where components have different lead times. As an adjustment, for components with lead time $L_k > L_1$, we raise base stock levels from $\mathbf{y}^{k*}$ by an amount that equals the mean demand over a period of length $L_k - L_1$,

$$(L_k - L_1)A^k\boldsymbol{\mu} = A^k\mathbf{E}[\mathbf{D}(t - L_k, t - L_1)], \quad 1 < k \le K.$$

To show that our IBS policy is well-defined, we refer to Theorem 2 in [6], which states that there exists a finite constant M such that for any optimal solution of (4), $\mathbf{y}^* = (y_1^*, \ldots, y_n^*)'$, $|y_j^*| \le M$, $1 \le j \le n$. Therefore $\mathbf{y}^*$ are finite values that can be used to set base stock levels.

4. Performance analysis

We evaluate our IBS policy in two steps. In Section 4.1, we use $\tilde{\Phi}$, the optimal objective value of (4), to construct a lower bound on the average cost $\mathcal{C}$. In Section 4.2, we develop a novel bound on the optimality gap between the inventory cost under our IBS policy and the cost lower bound.

4.1. Lower bound

Under a feasible inventory policy, backlog and inventory levels are nonnegative at all times. Replenishment and allocation policies are only based on information available when decisions are made. For the family of policies that satisfies the above conditions, Reiman and Wang [5] show that

$$\underline{C} \equiv \mathbf{b} \cdot \sum_{k=1}^{K} \mathbf{E}[\mathbf{D}^k] + \inf_{\boldsymbol{\alpha} \in \mathbb{R}_m^+} \{\mathbf{b} \cdot \boldsymbol{\alpha} + \Phi^K(\boldsymbol{\alpha})\} \tag{6}$$

is a lower bound on the inventory cost (2), where $\Phi^K(\boldsymbol{\alpha})$ is the optimal objective value of the following $K + 1$ stage stochastic program (SP)

$$\begin{aligned} \Phi^K(\boldsymbol{\alpha}) &= \inf_{\mathbf{y}^K \ge 0} \{\mathbf{h}^K \cdot \mathbf{y}^K + \mathbf{E}[\Phi^{K-1}(\mathbf{y}^K, \boldsymbol{\alpha} + \mathbf{D}^K)]\}, \\ \Phi^k(\mathbf{y}^{k+1}, \ldots, \mathbf{y}^K, \mathbf{x}) &= \inf_{\mathbf{y}^k \ge 0} \{\mathbf{h}^k \cdot \mathbf{y}^k + \mathbf{E}[\Phi^{k-1}(\mathbf{y}^k, \ldots, \mathbf{y}^K, \mathbf{x} + \mathbf{D}^k)]\}, \\ & \qquad k = 1, \ldots, K - 1, \\ \Phi^0(\mathbf{y}^1, \ldots, \mathbf{y}^K, \mathbf{x}) &= -\max_{\mathbf{z} \ge 0} \{\mathbf{c} \cdot \mathbf{z} | \mathbf{z} \le \mathbf{x}, A\mathbf{z} \le (\mathbf{y}^1, \ldots, \mathbf{y}^K)'\}. \end{aligned} \tag{7}$$

To put this formulation into perspective, the SP imitates a myopic minimization of the expected inventory cost at a given time t in an ATO system. Determining $\mathbf{y}^k$ at stage k corresponds to choosing inventory levels of components with lead time L_k at time $t - L_k$ $(1 \le k \le K)$. Determining $\mathbf{z}$ in the last stage LP corresponds to optimizing component allocation to serve demands for different products. In addition to these decisions, backlogs at $t - L_K$, denoted by $\boldsymbol{\alpha}$, also affect the inventory cost at time t. A larger $\boldsymbol{\alpha}$ always helps (in the weak sense): at worst, they can be removed by increasing inventory levels by $A\boldsymbol{\alpha}$ to keep the inventory cost intact, while the additional backlog may provide more flexibility for reducing the inventory cost. To set a lower bound, we may need some components of $\boldsymbol{\alpha}$ to go to infinity. In any case,

$$\underline{C} = \mathbf{b} \cdot \sum_{k=1}^{K} \mathbf{E}[\mathbf{D}^k] + \lim_{\boldsymbol{\alpha} \to \infty} \{\mathbf{b} \cdot \boldsymbol{\alpha} + \Phi^K(\boldsymbol{\alpha})\}. \tag{8}$$

The following theorem uses $\tilde{\Phi}$ to construct a more relaxed cost lower bound than $\underline{C}$.

Theorem 1.

$$\underline{C} \ge \underline{C}^* \equiv \mathbf{b} \cdot \mathbf{E}[\mathbf{D}^1] + \tilde{\Phi}. \tag{9}$$

Proof. Let $\Phi(\boldsymbol{\alpha})$ be the optimal solution of the following two-stage SP

$$\left.\begin{aligned} \Phi(\boldsymbol{\alpha}) &= \inf_{\mathbf{y} \ge 0} \{\mathbf{h} \cdot \mathbf{y} + \mathbf{E}[\Phi^0(\mathbf{y}, \boldsymbol{\alpha} + \mathbf{D}^1)]\} \\ \text{where } \Phi^0(\mathbf{y}, \boldsymbol{\alpha} + \mathbf{D}^1) &= -\max_{\mathbf{z} \ge 0} \{\mathbf{c} \cdot \mathbf{z} | \mathbf{z} \le \boldsymbol{\alpha} + \mathbf{D}^1, A\mathbf{z} \le \mathbf{y}\}, \end{aligned}\right\} \tag{10}$$

and $\boldsymbol{\alpha} \in \mathbb{R}_m^+$. Then

$$\underline{C}^* = \mathbf{b} \cdot \mathbf{E}[\mathbf{D}^1] + \lim_{\boldsymbol{\alpha} \to \infty} \{\mathbf{b} \cdot \boldsymbol{\alpha} + \Phi(\boldsymbol{\alpha})\}, \tag{11}$$

as by Theorem 2 in [6], $\tilde{\Phi} = \lim_{\boldsymbol{\alpha} \to \infty} \{\mathbf{b} \cdot \boldsymbol{\alpha} + \Phi(\boldsymbol{\alpha})\}$.

To bridge the gap between $\Phi^K(\boldsymbol{\alpha})$ in (7) and $\Phi(\boldsymbol{\alpha})$ in (10), we define below an auxiliary SP

$$\begin{aligned} &\psi^K(\boldsymbol{\alpha}) = \inf_{\mathbf{y}^K \ge 0} \{\mathbf{h}^K \cdot \mathbf{y}^K + \psi^{K-1}(\mathbf{y}^K, \boldsymbol{\alpha} + \mathbf{E}[\mathbf{D}^K])\}, \\ &\psi^k(\mathbf{y}^{k+1}, \ldots, \mathbf{y}^K, \mathbf{x}) \\ &\quad = \inf_{\mathbf{y}^k \ge 0} \{\mathbf{h}^k \cdot \mathbf{y}^k + \psi^{k-1}(\mathbf{y}^k, \ldots, \mathbf{y}^K, \mathbf{x} + \mathbf{E}[\mathbf{D}^k])\}, \\ &k = K - 1, \ldots, 2, \\ &\psi^1(\mathbf{y}^2, \ldots, \mathbf{y}^K, \mathbf{x}) = \Phi^1(\mathbf{y}^2, \ldots, \mathbf{y}^K, \mathbf{x}) \\ &\quad = \inf_{\mathbf{y}^1 \ge 0} \{\mathbf{h}^1 \cdot \mathbf{y}^1 + \mathbf{E}[\psi^0(\mathbf{y}^1, \ldots, \mathbf{y}^K, \mathbf{x} + \mathbf{D}^1)]\}, \\ &\psi^0(\mathbf{y}^1, \ldots, \mathbf{y}^K, \mathbf{x}) = \Phi^0(\mathbf{y}^1, \ldots, \mathbf{y}^K, \mathbf{x}) \\ &\quad = -\max_{\mathbf{z} \ge 0} \{\mathbf{c} \cdot \mathbf{z} | \mathbf{z} \le \mathbf{x}, A\mathbf{z} \le (\mathbf{y}^1, \ldots, \mathbf{y}^K)'\}. \end{aligned}$$

Applying the above to expand $\psi^K(\boldsymbol{\alpha})$ progressively,

$$\begin{aligned} \psi^K(\boldsymbol{\alpha}) &= \inf_{\mathbf{y}^1, \ldots, \mathbf{y}^K \ge 0} \left\{ \sum_{k=1}^{K} \mathbf{h}^k \cdot \mathbf{y}^k + \mathbf{E}[\psi^0(\mathbf{y}^1, \ldots, \mathbf{y}^K, \boldsymbol{\alpha} \right. \\ &\qquad \left. + \mathbf{E}[\mathbf{D}^K + \cdots + \mathbf{D}^2] + \mathbf{D}^1)] \right\} \\ &= \Phi\left(\boldsymbol{\alpha} + \mathbf{E}[\mathbf{D}^K + \cdots + \mathbf{D}^2]\right). \end{aligned}$$

It follows that

$$\lim_{\boldsymbol{\alpha} \to \infty} (\mathbf{b} \cdot \boldsymbol{\alpha} + \Phi(\boldsymbol{\alpha})) = \mathbf{b} \cdot \sum_{k=2}^{K} \mathbf{E}[\mathbf{D}^k] + \lim_{\boldsymbol{\alpha} \to \infty} \left(\mathbf{b} \cdot \boldsymbol{\alpha} + \psi^K(\boldsymbol{\alpha})\right).$$

Using this condition, along with (8) and (11), we prove the theorem by showing next that

$$\Phi^K(\boldsymbol{\alpha}) \ge \psi^K(\boldsymbol{\alpha}) \quad \text{for all } \boldsymbol{\alpha} \in \mathbb{R}_m^+. \tag{12}$$

Note that for $k = 0$, the LP objective value $\Phi^k(\mathbf{y}^{k+1}, \ldots, \mathbf{y}^K, \mathbf{x})$ is convex in $\mathbf{x}$ (cf. [1, p. 213]). Because convexity is preserved after taking the infimum and the expectation, the objective value $\Phi^k(\mathbf{y}^{k+1}, \ldots, \mathbf{y}^K, \mathbf{x})$ is also convex in $\mathbf{x}$ when $k = 1, \ldots, K$.

By definition, when $k = 0$ or 1,

$$\Phi^k(\mathbf{y}^{k+1}, \ldots, \mathbf{y}^K, \mathbf{x}) \ge \psi^k(\mathbf{y}^{k+1}, \ldots, \mathbf{y}^K, \mathbf{x}).$$

To carry out an induction, assume it is true for some $k \ge 1$, then

$$\begin{aligned} &\Phi^{k+1}(\mathbf{y}^{k+2}, \ldots, \mathbf{y}^K, \mathbf{x}) \\ &\quad = \inf_{\mathbf{y}^{k+1} \ge 0} \{\mathbf{h}^{k+1} \cdot \mathbf{y}^{k+1} + \mathbf{E}[\Phi^k(\mathbf{y}^{k+1}, \ldots, \mathbf{y}^K, \mathbf{x} + \mathbf{D}^{k+1})]\} \\ &\quad \ge \inf_{\mathbf{y}^{k+1} \ge 0} \{\mathbf{h}^{k+1} \cdot \mathbf{y}^{k+1} + \Phi^k(\mathbf{y}^{k+1}, \ldots, \mathbf{y}^K, \mathbf{x} + \mathbf{E}[\mathbf{D}^{k+1}])\} \\ &\quad \ge \inf_{\mathbf{y}^{k+1} \ge 0} \{\mathbf{h}^{k+1} \cdot \mathbf{y}^{k+1} + \psi^k(\mathbf{y}^{k+1}, \ldots, \mathbf{y}^K, \mathbf{x} + \mathbf{E}[\mathbf{D}^{k+1}])\} \\ &\quad = \psi^{k+1}(\mathbf{y}^{k+2}, \ldots, \mathbf{y}^K, \mathbf{x}), \end{aligned}$$

where the first inequality follows from Jensen's Inequality and the second one from the induction hypothesis. This completes the induction.

4.2. Optimality gap

For the lower bound SP (4), define

$$\mathbf{Q} = A\mathbf{D}^1 - \mathbf{y}^* \tag{13}$$

as the component balance (i.e., demand – supply). Component j has shortage when $Q_j > 0$ $(1 \le j \le n)$, in which case backlogs become inevitable. Given $\mathbf{Q}$, the optimal backlog levels are

$$\min_{\mathbf{B}\ge 0}\{\mathbf{c}\cdot\mathbf{B} | A\mathbf{B} \ge \mathbf{Q}\}, \tag{14}$$

which is equivalent to the second-stage LP (5). Let $\mathbf{z}^*$ be an optimal solution of the latter LP. Then $\mathbf{B}^* = \mathbf{D}^1 - \mathbf{z}^*$ is an optimal solution of (14). The reverse is also true.

Parallel to (13), define

$$\mathbf{Q}^k(t) = A^k\mathbf{B}(t) - \mathbf{I}^k(t), \quad 1 \le k \le K,\ t \ge 0,$$

as the component balance in the ATO system. Observe that by the definition of our IBS policy,

$$\mathbf{Y}^k = \mathbf{y}^{k*} + A^k\boldsymbol{\mu}(L_k - L_1) \quad \text{and}$$
$$\mathbf{Y}^k = \mathbf{I}^k(t-L_k) + \mathbf{R}^k(t-L_k) - A^k\mathbf{B}(t-L_k), \quad 1 \le k \le K,\ t \ge 0.$$

Applying the above and (1) to eliminate $\mathbf{B}(t)$ and $\mathbf{I}^k(t)$ $(1 \le k \le K)$, balances of components with lead time k $(1 \le k \le K)$ can also be expressed as

$$\mathbf{Q}^k(t) = A^k\mathbf{D}^1(t) - \mathbf{y}^{k*} + A^k\sum_{l=2}^{k}\left(\mathbf{D}^l(t) - \mathbf{E}[\mathbf{D}^l(t)]\right), \quad t \ge 0. \tag{15}$$

Lemma 1 below shows that the excess of the expected inventory cost of an ATO system over its SP-based lower bound is completely reflected by the difference of backlogs between the two cases.

Lemma 1. *Let $\mathcal{C}(t)$ be the expected inventory cost under our IBS replenishment policy and any feasible allocation policy at time t $(t \ge L_K)$ and $\underline{C}^*$ be the lower bound given in (9). Then*

$$\mathcal{C}(t) - \underline{C}^* = \mathbf{c}\cdot\left(\mathbf{E}[\mathbf{B}(t)] - \mathbf{E}[\mathbf{B}^*]\right). \tag{16}$$

Proof. By definition, given base-stock levels $\mathbf{Y}^k$,

$$\mathbf{I}^k(t-L_k) + \mathbf{R}^k(t-L_k) = \mathbf{Y}^k + A^k\mathbf{B}(t-L_k), \quad 1 \le k \le K.$$

Apply the above and (1), the expected inventory cost at time t is

$$\mathcal{C}(t) = \sum_{k=1}^{K}\mathbf{h}^k\cdot\mathbf{E}[\mathbf{I}^k(t)] + \mathbf{b}\cdot\mathbf{E}[\mathbf{B}(t)]$$
$$= \mathbf{c}\cdot\mathbf{E}[\mathbf{B}(t)] + \sum_{k=1}^{K}\mathbf{h}^k\cdot\left(\mathbf{Y}^k - \mathbf{E}[A^k\mathbf{D}(t-L_k,t)]\right).$$

Let $\mathbf{z}^*$ be the optimal solution of $\tilde{\Phi}^0(\mathbf{y}^*, \mathbf{D}^1)$ in (5). Define $\mathbf{B}^* = \mathbf{D}^1 - \mathbf{z}^*$. Then $\mathbf{B}^*$ is the optimal solution of (14) and

$$\underline{C}^* = \mathbf{b}\cdot\mathbf{E}[\mathbf{D}^1] + \sum_{k=1}^{K}\mathbf{h}^k\cdot\mathbf{y}^{k*} - \mathbf{c}\cdot\mathbf{E}[\mathbf{z}^*]$$
$$= \sum_{k=1}^{K}\mathbf{h}^k\cdot\mathbf{y}^{k*} + \mathbf{c}\cdot\mathbf{E}[\mathbf{B}^*] - \sum_{k=1}^{K}\mathbf{h}^k\cdot\mathbf{E}[A^k\mathbf{D}^1].$$

The lemma follows because by (3),

$$\mathbf{Y}^k - \mathbf{E}[A^k\mathbf{D}(t-L_k,t)] = \mathbf{y}^{k*} - \mathbf{E}[A^k\mathbf{D}^1], \quad 1 \le k \le K. \quad \blacksquare$$

Observe that $\mathbf{E}[\mathbf{B}(t)]$ and $\mathbf{E}[\mathbf{B}^*]$ can differ for two reasons. First, (13) and (15) imply that component balances under our IBS policy may differ in distribution from those in the SP. Second, given component balance, the allocation policy of ATO systems may not achieve the perfect allocation outcome of (14). To separate these two influences, we define $\mathbf{B}^*(t)$ as the optimal solution of

$$\min_{\mathbf{B}\ge 0}\{\mathbf{c}\cdot\mathbf{B} | A^k\mathbf{B} \ge \mathbf{Q}^k(t),\ 1 \le k \le K\}, \tag{17}$$

and divide the relative difference of the inventory cost from its lower bound into two parts:

$$\frac{\mathcal{C}(t) - \underline{C}^*}{\underline{C}^*} = \frac{\mathbf{c}\cdot(\mathbf{E}[\mathbf{B}^*(t)] - \mathbf{E}[\mathbf{B}^*])}{\underline{C}^*} + \frac{\mathbf{c}\cdot(\mathbf{E}[\mathbf{B}(t)] - \mathbf{E}[\mathbf{B}^*(t)])}{\underline{C}^*}. \tag{18}$$

Assuming perfect allocation in the ATO system, the first part,

$$\frac{\mathbf{c}\cdot(\mathbf{E}[\mathbf{B}^*(t)] - \mathbf{E}[\mathbf{B}^*])}{\underline{C}^*}, \tag{19}$$

isolates the effect of different component balances under our base-stock policy and the SP solution. For a given balance process induced by our IBS policy, the second part,

$$\frac{\mathbf{c}\cdot(\mathbf{E}[\mathbf{B}(t)] - \mathbf{E}[\mathbf{B}^*(t)])}{\underline{C}^*}, \tag{20}$$

measures the cost difference from that of the perfect allocation. To evaluate IBS policies, we focus on (19), but will discuss briefly an allocation policy for minimizing (20) in the next section.

Define $\mathbb{D}^k = \mathbf{D}^2 + \cdots + \mathbf{D}^k$. Notice that for $j = 1, \ldots, n$, $\mathbf{A}_j \cdot \mathbb{D}^{k_j}$ represents the amount of component j needed to serve all demand arriving during a period of length $L_{k_j} - L_1$.

Lemma 2. *There exists a finite constant κ_1, depending only on values of $\mathbf{c}$ and A, such that*

$$\mathbf{c}\cdot\left(\mathbf{E}[\mathbf{B}^*(t)] - \mathbf{E}[\mathbf{B}^*]\right) \le \kappa_1\mathbf{E}\left[\sum_{j=1}^{n}\left|\mathbf{A}_j\cdot\mathbb{D}^{k_j} - \mathbf{E}[\mathbf{A}_j\cdot\mathbb{D}^{k_j}]\right|\right]. \tag{21}$$

Proof. Since $\mathbf{D}^1(t) \overset{d}{=} \mathbf{D}^1$, we can compare $\mathbf{B}^*(t)$ and $\mathbf{B}^*$ on matching sample paths $(\mathbf{D}^1(t) = \mathbf{D}^1)$. From (14) and (17), $\mathbf{B}^*$ and $\mathbf{B}^*(t)$ are optimal solutions of the same LP with different RHS values. Following standard LP sensitivity analysis [8], there exists a finite constant κ_1 such that

$$\mathbf{c}\cdot\left(\mathbf{B}^*(t) - \mathbf{B}^*\right) \le \kappa_1\max_{1\le j\le n}|Q_j(t) - Q_j|.$$

It follows that

$$\mathbf{c}\cdot\left(\mathbf{E}[\mathbf{B}^*(t)] - \mathbf{E}[\mathbf{B}^*]\right) \le \kappa_1\mathbf{E}\left[\max_{1\le j\le n}|Q_j(t) - Q_j|\right]$$
$$= \kappa_1\mathbf{E}\left[\max_{1\le j\le n}\left|\sum_{k=2}^{k_j}\mathbf{A}_j\cdot\mathbf{D}^k(t) - \mathbf{E}\left[\sum_{k=2}^{k_j}\mathbf{A}_j\cdot\mathbf{D}^k(t)\right]\right|\right]$$
$$= \kappa_1\mathbf{E}\left[\max_{1\le j\le n}\left|\mathbf{A}_j\cdot\mathbb{D}^{k_j} - \mathbf{E}\left[\mathbf{A}_j\cdot\mathbb{D}^{k_j}\right]\right|\right]$$
$$\le \kappa_1\mathbf{E}\left[\sum_{j=1}^{n}\left|\mathbf{A}_j\cdot\mathbb{D}^{k_j} - \mathbf{E}\left[\mathbf{A}_j\cdot\mathbb{D}^{k_j}\right]\right|\right],$$

where the first equality comes from (13) and (15). $\blacksquare$

Lemma 3. *There exists a finite constant κ_2, depending only on values of $\mathbf{b}$, $\mathbf{h}$, and A, such that*

$$\underline{C}^* \ge \kappa_2\sum_{j=1}^{n}\mathbf{E}\left[\left|\mathbf{A}_j\cdot\mathbf{D}^1 - \mathbf{E}[\mathbf{A}_j\cdot\mathbf{D}^1]\right|\right]. \tag{22}$$

Proof. From (9) and (4),

$$\underline{C}^* = \sum_{i=1}^m b_i \mathbf{E}[D_i^1 - z_i^*] + \sum_{j=1}^n h_j \mathbf{E}[y_j^* - \mathbf{A}_j \cdot \mathbf{z}^*].$$

Let $\tilde{b} = \min_{1\le i\le m}\left\{\frac{b_i}{\sum_{j=1}^n a_{ji}}\right\}$. Then

$$\sum_{i=1}^m b_i(D_i^1 - z_i^*) \ge \tilde{b}\sum_{i=1}^m\sum_{j=1}^n a_{ji}(D_i^1 - z_i^*)$$

$$= \tilde{b}\sum_{j=1}^n (\mathbf{A}_j \cdot \mathbf{D}^1 - \mathbf{A}_j\cdot\mathbf{z}^*).$$

Since $\mathbf{z}^* \le \mathbf{D}^1$ and $\mathbf{A}_j \cdot \mathbf{z}^* \le y_j^*$ $(1 \le j \le n)$,

$$\underline{C}^* \ge \sum_{j=1}^n \left\{\tilde{b}\mathbf{E}\left[(\mathbf{A}_j\cdot\mathbf{D}^1 - y_j^*)^+\right] + h_j\mathbf{E}\left[(y_j^* - \mathbf{A}_j\cdot\mathbf{D}^1)^+\right]\right\}.$$

Define $F_j(x)$ as the CDF of $\mathbf{A}_j \cdot \mathbf{D}^1$ and $\bar{F}_j(x) = 1 - F_j(x)$ $(1 \le j \le n)$. Let

$$q_j = \frac{\bar{F}_j(\mathbf{E}[\mathbf{A}_j\cdot\mathbf{D}^1])}{F_j(\mathbf{E}[\mathbf{A}_j\cdot\mathbf{D}^1])}, \quad 1 \le j \le n.$$

Let $\underline{b}_j = \tilde{b} \wedge (h_j q_j)$ and $\underline{h}_j = h_j \wedge (\tilde{b}/q_j)$, $1 \le j \le n$. Then

$$\underline{b}_j \le \tilde{b}, \qquad \underline{h}_j \le h_j, \quad \text{and} \quad \bar{F}_j(\mathbf{E}[\mathbf{A}_j\cdot\mathbf{D}^1]) = \frac{\underline{b}_j}{\underline{b}_j + \underline{h}_j},$$
$$1 \le j \le n.$$

Hence $\mathbf{E}[\mathbf{A}_j \cdot \mathbf{D}^1]$ is the optimal solution of the Newsvendor model

$$\min_x \left\{\underline{b}_j\mathbf{E}[(\mathbf{A}_j\cdot\mathbf{D}^1 - x)^+] + \underline{h}_j\mathbf{E}[(x - \mathbf{A}_j\cdot\mathbf{D}^1)^+]\right\}, \quad 1 \le j \le n,$$

and it follows that

$$\begin{aligned}\underline{C}^* &\ge \sum_{j=1}^n \left\{\underline{b}_j\mathbf{E}[(\mathbf{A}_j\cdot\mathbf{D}^1 - y_j^*)^+] + \underline{h}_j\mathbf{E}[(y_j^* - \mathbf{A}_j\cdot\mathbf{D}^1)^+]\right\} \\ &\ge \sum_{j=1}^n \left\{\underline{b}_j\mathbf{E}[(\mathbf{A}_j\cdot\mathbf{D}^1 - \mathbf{E}[\mathbf{A}_j\cdot\mathbf{D}^1])^+]\right. \\ &\quad \left. + \underline{h}_j\mathbf{E}[(\mathbf{E}[\mathbf{A}_j\cdot\mathbf{D}^1] - \mathbf{A}_j\cdot\mathbf{D}^1)^+]\right\} \\ &\ge \sum_{j=1}^n (\underline{b}_j \wedge \underline{h}_j)\mathbf{E}[|\mathbf{A}_j\cdot\mathbf{D}^1 - \mathbf{E}[\mathbf{A}_j\cdot\mathbf{D}^1]|]. \qquad (23)\end{aligned}$$

The lemma follows by defining $\kappa_2 \equiv \min_{1\le j\le n}\left(\underline{b}_j \wedge \underline{h}_j\right)$. ■

Based on above lemmas, we now compare inventory costs between the ATO system that follows our replenishment policy under perfect allocation and that of the corresponding SP (19). The theorem below establishes an upper bound on the cost difference.

Theorem 2. *There exist a threshold $\bar{L}$ and finite constants $\beta_1, \beta_2 > 0$ that do not depend on t such that*

$$\frac{\mathbf{c}\cdot(\mathbf{E}[\mathbf{B}^*(t)] - \mathbf{E}[\mathbf{B}^*])}{\underline{C}^*} \le \beta_1\left(\frac{L_K}{L_1} - 1\right), \qquad (24)$$

if $L_1 \le \bar{L}$. Otherwise

$$\frac{\mathbf{c}\cdot(\mathbf{E}[\mathbf{B}^*(t)] - \mathbf{E}[\mathbf{B}^*])}{\underline{C}^*} \le \beta_2\sqrt{\frac{L_K}{L_1} - 1}. \qquad (25)$$

Proof. Let $\mathbf{D}$ be a random vector such that $\mathbf{D} \overset{d}{=} \mathcal{D}(1)$. Define $X_j^0 = \mathbf{A}_j \cdot \mathbf{D}$ as the demand for component j $(1 \le j \le n)$ within a unit interval, which has a finite mean $\mathbf{E}[X_j^0]$ and a finite standard deviation $\sigma_{X_j^0}$. Define $X_j \equiv \mathbf{A}_j\cdot\mathbb{D}^{k_j}$, $1 \le j \le n$. Since X_j is positive and compound Poisson, its mean absolute deviation (MAD) satisfies

$$\mathbf{E}\left[\left|X_j - \mathbf{E}[X_j]\right|\right] \le 2\mathbf{E}[X_j] = 2(L_{k_j} - L_1)\mathbf{E}[X_j^0], \quad 1 \le j \le n. \qquad (26)$$

The MAD is also bounded by the standard deviation, i.e.,

$$\mathbf{E}\left[\left|X_j - \mathbf{E}[X_j]\right|\right] \le \sigma_{X_j} = \sigma_{X_j^0}\sqrt{L_{k_j} - L_1}, \quad 1 \le j \le n. \qquad (27)$$

We apply (26) and (27) to (21) to bound $\mathbf{c}\cdot(\mathbf{E}[\mathbf{B}^*(t)] - \mathbf{E}[\mathbf{B}^*])$. Define $W_j \equiv \mathbf{A}_j \cdot \mathbf{D}^1$ $(1 \le j \le n)$, which is compound Poisson. Denote $\Pr(\Lambda(L_1) = l) = p_l$ as the probability that the number of orders during $[0, L_1]$ is l $(l = 0, 1, \ldots)$ and $\lambda_1 \equiv E[\Lambda(L_1)] = \underline{\lambda}L_1$ (where $\underline{\lambda}$ is the order arrival rate).

Given $\mathbf{S} = (S_1, \ldots, S_m)$ as a generic element of the order size sequence where S_i is the order size of product i $(1 \le i \le m)$, the order size of component j is $s_j = \mathbf{A}_j \cdot \mathbf{S}$, which has a finite mean $\bar{s}_j$ and a finite standard deviation σ_{s_j} $(1 \le j \le n)$. It follows that

$$\mathbf{E}[W_j] = \lambda_1\bar{s}_j \quad \text{and} \quad \sigma_{W_j} = \sigma_{s_j}\sqrt{\lambda_1}, \quad 1 \le j \le n.$$

Let

$$\upsilon_j = (\sqrt{2} + 1)\frac{\sigma_{s_j} + 1}{\bar{s}_j}, \quad 1 \le j \le n.$$

Let $\mathcal{P}(x)$ be the CDF of standard normal distribution. For a Poisson variable Λ with mean λ,

$$\begin{aligned}\Pr(\Lambda \le \lfloor\lambda - \upsilon_j\sqrt{\lambda}\rfloor) &= \Pr(\Lambda \le \lambda - \upsilon_j\sqrt{\lambda}) \\ &= \Pr\left(\frac{\Lambda - \lambda}{\sqrt{\lambda}} \le -\upsilon_j\right) \longrightarrow \mathcal{P}(-\upsilon_j) \quad \text{as } \lambda \longrightarrow \infty, 1 \le j \le n.\end{aligned}$$

The first equation holds because Λ takes only integer values, and the convergence follows from the Central Limit Theorem. Thus there exists some $\bar{\lambda}_j$ such that for all $\lambda \ge \bar{\lambda}_j$,

$$\Pr(\Lambda \le \lfloor\lambda - \upsilon_j\sqrt{\lambda}\rfloor) \ge 0.8\mathcal{P}(-\upsilon_j), \quad 1 \le j \le n.$$

Define $u_j = \bar{\lambda}_j \vee \lceil\upsilon_j^2\rceil$, $1 \le j \le n$, which do not depend on L_1. Define $\bar{L} \equiv \min_{1\le j\le n}\{u_j/\underline{\lambda}\}$. When $L_1 \le \bar{L}$, $\lambda_1 = \underline{\lambda}L_1 \le u_j$ for all j. Therefore

$$\mathbf{E}[|W_j - \mathbf{E}[W_j]|] \ge p_0\mathbf{E}[W_j] \ge e^{-u_j}\lambda_1\bar{s}_j = e^{-u_j}\underline{\lambda}L_1\bar{s}_j,$$
$$1 \le j \le n. \qquad (28)$$

To prove (24), use Lemma 2 and (26) to set an upper bound on the numerator below on the left-hand side. Use Lemma 3 and (28) to set a lower bound on the denominator. Then there exists a finite constant β_1 that does not depend on t such that

$$\frac{\mathbf{c}\cdot(\mathbf{E}[\mathbf{B}^*(t)] - \mathbf{E}[\mathbf{B}^*])}{\underline{C}^*} \le \beta_1\left(\frac{L_K}{L_1} - 1\right). \qquad (29)$$

When $L_1 > \bar{L}$, $\lambda_1 \ge u_j$ for some j. Fix such a component j, and let $\xi_j = (\sigma_{s_j} + 1)$. Then for $l = 1, \ldots, \lfloor\lambda_1 - \upsilon_j\sqrt{\lambda_1}\rfloor$, $l \ge 0$ because $\sqrt{\lambda_1} \ge \sqrt{u_j} \ge \upsilon_j$, and

$$(\lambda_1 - l)\bar{s}_j - \xi_j\sqrt{\lambda_1} \ge (\upsilon_j\bar{s}_j - \xi_j)\sqrt{\lambda_1} = \sqrt{2}(\sigma_{s_j} + 1)\sqrt{\lambda_1}. \qquad (30)$$

Let s_j^i $(i = 1, \dots,)$ be i.i.d. copies of s_j. Then for all $l = 1, \dots, \lfloor \lambda_1 - \upsilon_j \sqrt{\lambda_1} \rfloor$,

$$\begin{aligned} &\Pr\left(\sum_{i=1}^{l} s_j^i > \mathbf{E}[W_j] - \xi_j\sqrt{\lambda_1}\right) \\ &= \Pr\left(\sum_{i=1}^{l} s_j^i - l\bar{s}_j > (\lambda_1 - l)\bar{s}_j - \xi_j\sqrt{\lambda_1}\right) \\ &\le \frac{l\sigma_{s_j}^2}{[(\lambda_1 - l)\bar{s}_j - \xi_j\sqrt{\lambda_1}]^2} \quad \text{(Chebyshev Inequality)} \\ &< \frac{1}{2} \quad \text{(because of (30) and that } l < \lambda_1\text{).} \end{aligned} \tag{31}$$

It follows that

$$\begin{aligned} \mathbf{E}[|W_j - \mathbf{E}[W_j]|] &= \mathbf{E}\left[\sum_{l=0}^{\infty} p_l \left|\sum_{i=1}^{l} s_l^i - \lambda_1 \bar{s}_j\right|\right] \\ &\ge \sum_{l=0}^{\lfloor \lambda_1 - \upsilon_j\sqrt{\lambda_1}\rfloor} p_l \mathbf{E}\left[\left|\sum_{i=1}^{l} s_j^i - \lambda_1 \bar{s}_j\right|\right] \\ &\ge \xi_j\sqrt{\lambda_1} \sum_{l=0}^{\lfloor \lambda_1 - \upsilon_j\sqrt{\lambda_1}\rfloor} p_l \Pr\left(\sum_{i=1}^{l} s_j^i \le \lambda_1\bar{s}_j - \xi_j\sqrt{\lambda_1}\right) \\ &\ge \frac{\xi_j\sqrt{\lambda_1}}{2} \Pr\left(\Lambda(L_1) \le \lambda_1 - \upsilon_j\sqrt{\lambda_1}\right) \\ &\quad \text{(because of (31))} \\ &\ge 0.4\xi_j\sqrt{\underline{\lambda} L_1}\mathcal{P}(-\upsilon_j), \quad \text{(because } \lambda_1 \ge u_j\text{).} \end{aligned}$$

To prove (25), use the above and Lemma 3 to set a lower bound on $\underline{C}^*$. Use Lemma 2 and (27) to set an upper bound on the numerator on the left hand side below. Then there exists a finite constant $\beta_2 > 0$ that does not depend on t such that

$$\frac{\mathbf{c} \cdot (\mathbf{E}[\mathbf{B}^*(t)] - \mathbf{E}[\mathbf{B}^*])}{\underline{C}^*} \le \beta_2 \left(\sqrt{\frac{L_K}{L_1} - 1}\right). \quad \blacksquare$$

5. Component allocation and asymptotic optimality

Consider an ATO system with lead times $\mathbf{L} = (L_1, \dots, L_K)$ where $L_1 < \cdots < L_K$. Denote the cost objective in (2) by $\mathcal{C}^{(\mathbf{L})}$ and its lower bound in (9) by $\underline{C}^{*(\mathbf{L})}$. Let $\mathbf{B}^{(\mathbf{L})}(t)$ be the backlog levels at time t and $\mathbf{B}^{*(\mathbf{L})}(t)$ be the values defined in (17) $(t \ge 0)$. The following corollary to Theorem 2 shows that under perfect allocation, the percentage difference of the cost objective under our IBS policy from its lower bound diminishes to zero as the ratio of the longest and the shortest lead times converges to unity.

Corollary 1. *If the allocation policy results in*

$$\mathbf{c} \cdot \mathbf{E}[\mathbf{B}^{(\mathbf{L})}(t)] = \mathbf{c} \cdot \mathbf{E}[\mathbf{B}^{*(\mathbf{L})}(t)], \quad t \ge 0,$$

then under our IBS replenishment policy,

$$\lim_{L_K/L_1 \to 1} \frac{\mathcal{C}^{(\mathbf{L})} - \underline{C}^{*(\mathbf{L})}}{\underline{C}^{*(\mathbf{L})}} = 0.$$

Proof. Immediate from (18) and Theorem 2. ■

For ATO systems with an identical lead time L for all components, Reiman and Wang [6] develop the following allocation principle: let $\mathbf{Q}^{(L)}(t)$ be component balances at time t $(t \ge 0)$ (which they refer to as component shortage by considering any surplus inventory over existing demands as negative shortage). Set backlog targets at $\mathbf{B}^{*(L)}(t)$, an optimal solution of (17) selected to be Lipschitz continuous in $\mathbf{Q}^{(L)}(t)$. Any product with its backlog level currently at or below the target (i.e., $B_i^{(L)}(t) \le B_i^{*(L)}(t)$, $1 \le i \le m$) is not served. All other products are served to clear as much excess backlogs as possible, subject to component availability. They prove that under any allocation policy that satisfies this principle,

$$\lim_{L\to\infty} \sup_{t \ge L} \frac{\mathbf{c} \cdot \left[\mathbf{E}[\mathbf{B}^{(L)}(t)] - \mathbf{E}[\mathbf{B}^{*(L)}(t)]\right]}{\underline{C}^{*(L)}} = 0.$$

The following corollary extends asymptotic optimality of this allocation principle to our systems.

Corollary 2. *If, for some $\delta > 0$, the demand order size $\mathbf{S}$ has a finite moment of order $2+\delta$ and $L_K/L_1 \to 1$ as $L_1 \to \infty$, then under our IBS replenishment policy and an allocation policy that satisfies the above allocation principle,*

$$\frac{\mathcal{C}^{(\mathbf{L})} - \underline{C}^{*(\mathbf{L})}}{\underline{C}^{*(\mathbf{L})}} \to 0 \quad \textit{as } L_1 \to \infty.$$

Based on (18) and Theorem 2, we can prove the corollary by showing that

$$\lim_{L_1\to\infty} \sup_{t \ge L_K} \frac{\mathbf{c} \cdot (\mathbf{E}[\mathbf{B}^{(\mathbf{L})}(t)] - \mathbf{E}[\mathbf{B}^{*(\mathbf{L})}(t)])}{\underline{C}^{*(\mathbf{L})}} = 0, \tag{32}$$

which can be proven by a similar analysis as in Section 4 of [6]. To accommodate nonidentical lead times, we scale our systems by the shortest lead time L_1. The centered and scaled amount of demand arriving over a longer lead time L_k $(2 \le k \le K)$ satisfies

$$\begin{aligned} \frac{D_i(t - L_k, t) - L_k\mu_i}{\sqrt{L_1}} &= \Delta\hat{D}_i^{(L_1)}(t) + \hat{D}_i^{(L_1)}(t), \\ \text{where } \Delta\hat{D}_i^{(L_1)}(t) &= \frac{D_i(t - L_k, t - L_1) - (L_k - L_1)\mu_i}{\sqrt{L_1}}, \\ \hat{D}_i^{(L_1)}(t) &= \frac{D_i(t - L_1, t) - L_1\mu_i}{\sqrt{L_1}}, \quad 1 \le i \le m, \\ &t \ge 0. \end{aligned}$$

Because of stationarity of compound Poisson Processes, $\mathbf{E}[\Delta\hat{D}_i^{(L_1)}(t)]$ does not depend on t. Applying the Central Limit Theorem and the aforementioned finiteness of moments of order $2+\delta$, if $L_K/L_1 \to 1$ as $L_1 \to \infty$, then

$$\lim_{L_1\to\infty} \sup_{t\ge L_K} \mathbf{E}\left[\Delta\hat{D}_i^{(L_1)}(t)\right] = 0, \quad 1 \le i \le m.$$

This allows us to adopt the same proof of Theorem 4 in [6] to prove (32), with all additional terms arising from lead time differences eliminated by the above. Hence, for ATO systems with nonidentical lead times, our IBS policies, while not exactly optimal in general, are asymptotically optimal when the lead times grow but their differences do not grow as fast.

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References

[1] D. Bertsimas, J.N. Tsitsiklis, Introduction to Linear Optimization, Athena Scientific, Nashua, NH, 1997.
[2] M. Doğru, M. Reiman, Q. Wang, A stochastic programming based inventory policy for assemble-to-order systems with application to the W model, Oper. Res. 58 (2010) 849–864.
[3] L. Lu, J. Song, H. Zhang, Optimal and asymptotically optimal policies for assemble-to-order N–and W-systems, Nav. Res. Logist. 62 (2015) 617–645.

[4] Y. Lu, J. Song, Y. Zhao, No-holdback allocation rules for continuous-time assemble-to-order systems, Oper. Res. 58 (2010) 691–705.
[5] M.I. Reiman, Q. Wang, A stochastic program based lower bound for assemble-to-order inventory systems, Oper. Res. Lett. 40 (2012) 89–95.
[6] M.I. Reiman, Q. Wang, Asymptotically optimal inventory control of assemble-to-order systems with identical lead times, Oper. Res. 63 (2015) 716–732.
[7] K. Rosling, Optimal inventory policies for assembly systems under random demands, Oper. Res. 37 (1989) 565–579.
[8] A. Schrijver, Theory of Linear and Integer Programming, John Wiley & Sons, 1998.
[9] J. Song, P. Zipkin, Supply chain operations: Assemble-to-order systems, in: A. de Kok, S. Graves (Eds.), Handbooks in Operations Research and Management Science, Vol. 11, Elsevier, 2003, pp. 561–596.
[10] W. van Jaarsveld, A. Scheller-Wolf, Optimization of industrial-scale assemble-to-order systems, INFORMS J. Comput. 27 (2015) 544–560.
[11] P.H. Zipkin, Foundations of Inventory Management, McGraw-Hill, 2000.