Asymptotically Optimal Inventory Control for Assemble-to-Order Systems with Identical Lead Times E-Companion

Martin I. Reiman
Alcatel-Lucent Bell Labs, Murray Hill, NJ 07974
email: marty@research.bell-labs.com

Qiong Wang
Industrial and Enterprise Systems Engineering
University of Illinois at Urbana-Champaign, Urbana, IL 61801
qwang04@illinois.edu

Proofs

EC.1. Proof of Theorem 2

We first prove that if $\mathbf{y}^*$ optimizes (16), then

$$|y_j^*| \le M, \quad 1 \le j \le n. \tag{EC.1}$$

Suppose that in (16), given $\mathbf{y}$ and $\mathbf{D}$, $\mathbf{z}^*(\mathbf{y};\mathbf{D})$ optimizes $\varphi(\mathbf{y};\mathbf{D})$. Then the objective function is

$$C(\mathbf{y}) = \mathbf{h}\cdot E[\mathbf{y} - A\mathbf{z}^*(\mathbf{y};\mathbf{D})] + \mathbf{b}\cdot E[\mathbf{D} - \mathbf{z}^*(\mathbf{y};\mathbf{D})]. \tag{EC.2}$$

Denote $\mathbf{z}^*(\mathbf{y}^*;\mathbf{D})$ by $\mathbf{z}^*$. Since $A\mathbf{z}^* \le \mathbf{y}$ and $\mathbf{z}^* \le \mathbf{D}$,

$$\begin{aligned} C(\mathbf{y}) &\ge \mathbf{b}\cdot E[\mathbf{D}-\mathbf{z}^*] \ge \sum_{i:a_{ji}>0} a_{ji}\frac{b_i}{a_{ji}} E[D_i - z_i^*] \ge \zeta_j \mathbf{A}_j \cdot E[\mathbf{D}-\mathbf{z}^*] \ge \zeta_j(\mathbf{A}_j\cdot E[\mathbf{D}] - y_j), \\ \text{and } C(\mathbf{y}) &\ge \mathbf{h}\cdot E[\mathbf{y} - A\mathbf{z}^*] \ge h_j(y_j - \mathbf{A}_j\cdot E[\mathbf{z}^*]) \ge h_j(y_j - \mathbf{A}_j \cdot E[\mathbf{D}]). \end{aligned} \tag{EC.3}$$

Because $\mathbf{y} = \mathbf{0}$ and $\mathbf{z}(\mathbf{y};\mathbf{D}) = \mathbf{0}$ is feasible for (16) in which case $C(\mathbf{0}) = \mathbf{b}\cdot E[\mathbf{D}]$, the above inequalities indicate that a necessary condition for $C(\mathbf{y}) \le \mathbf{b}\cdot E[\mathbf{D}]$ is that

$$y_j \ge \mathbf{A}_j \cdot E[\mathbf{D}] - \frac{\mathbf{b}\cdot E[\mathbf{D}]}{\zeta_j} \text{ and } y_j \le \mathbf{A}_j\cdot E[\mathbf{D}] + \frac{\mathbf{b}\cdot E[\mathbf{D}]}{h_j}, \quad 1\le j\le n,$$

which cannot hold if $|y_j| > M$ for some $j = 1,..,n$. Thus to be optimal, $\mathbf{y}^*$ must satisfy (EC.1).

Apply the same logic to (12). Since $\mathbf{y} = \mathbf{0}$ and $\mathbf{z}(\mathbf{y};\mathbf{D}) = \mathbf{0}$ is feasible,

$$\mathbf{b}\cdot E[\mathbf{D}] \ge C_+(\mathbf{y}^o) \ge h_j(y_j^o - \mathbf{A}_j \cdot E[\mathbf{D}]), \quad 1 \le j \le n,$$

which implies that

$$y_j^o \le \mathbf{A}_j \cdot E[\mathbf{D}] + \frac{\mathbf{b}\cdot E[\mathbf{D}]}{h_j} \le M, \quad 1\le j\le n.$$

We now prove $C^* = \underline{C}$. By replacing $\mathbf{z}$ with $\mathbf{z}' + \boldsymbol{\alpha}$ and $\mathbf{y}$ with $\mathbf{y}' + A\boldsymbol{\alpha}$, we transform (14) into

$$\underline{C} = \inf_{\boldsymbol{\alpha}\geq 0} \left\{ \inf_{\mathbf{y}'\geq -A\boldsymbol{\alpha}} \{\mathbf{h}\cdot\mathbf{y}' + \mathbf{b}\cdot E[\mathbf{D}] - E[\phi'(\mathbf{y}',\boldsymbol{\alpha};\mathbf{D})]\} \right\} \tag{EC.4}$$

$$\text{where} \quad \phi'(\mathbf{y}',\boldsymbol{\alpha};\mathbf{D}) \equiv \max_{\mathbf{z}'\geq -\boldsymbol{\alpha}} \{\mathbf{c}\cdot\mathbf{z}' | \mathbf{z}' \leq \mathbf{D}, A\mathbf{z}' \leq \mathbf{y}'\}.$$

Since any feasible solution for (EC.4) is also feasible for (16), $C^* \leq \underline{C}$.

To prove $C^* \geq \underline{C}$, define

$$G_1(\mathbf{y},\boldsymbol{\alpha}) \equiv E[\phi'(\mathbf{y},\boldsymbol{\alpha};\mathbf{D})] = E\left[\max_{\mathbf{z}\geq -\boldsymbol{\alpha}} \{\mathbf{c}\cdot\mathbf{z} | \mathbf{z}\leq \mathbf{D}, A\mathbf{z}\leq\mathbf{y}\}\right],$$

$$G_2(\mathbf{y}) \equiv E[\varphi(\mathbf{y};\mathbf{D})] = E\left[\max_{\mathbf{z}\in\mathbf{R}^m} \{\mathbf{c}\cdot\mathbf{z} | \mathbf{z}\leq \mathbf{D}, A\mathbf{z}\leq\mathbf{y}\}\right].$$

Let $\boldsymbol{\alpha}^{(k)} = (k, ..., k)$ and denote, as a feasible solution to (EC.4),

$$y_j^{(k)} \equiv y_j^* \vee (-k \sum_{i=1}^m a_{ji}).$$

Since $\mathbf{y}^*$ is bounded by (EC.1), if k is sufficiently large, $\mathbf{y}^{(k)} = \mathbf{y}^*$. Thus we only need to prove that

$$\lim_{k\to\infty} G_1(\mathbf{y}^*, \boldsymbol{\alpha}^{(k)}) \geq G_2(\mathbf{y}^*), \tag{EC.5}$$

because if it holds, then

$$\underline{C} \leq \lim_{k\to\infty} \left\{ \mathbf{h}\cdot\mathbf{y}^* + \mathbf{b}\cdot E[\mathbf{D}] - G_1(\mathbf{y}^*,\boldsymbol{\alpha}^{(k)}) \right\} \leq \mathbf{h}\cdot\mathbf{y}^* + \mathbf{b}\cdot E[\mathbf{D}] - G_2(\mathbf{y}^*) = C^*.$$

Given that $\mathbf{z}^*$ is the optimal solution that yields $\varphi(\mathbf{y}^*;\mathbf{D})$, $z_i^* \geq -k$ $(1 \leq i \leq m)$ if

$$\mathbf{A}_j \cdot \mathbf{D} \leq y_j^* + k\underline{a}, \quad 1 \leq j \leq n. \tag{EC.6}$$

Otherwise, if there exists some i $(i = 1, .., m)$ such that $z_i^* < -k$, then under (EC.6) and because $\mathbf{z} \leq \mathbf{D}$, no capacity constraint that involves z_i^* is binding. Thus increasing z_i^* is feasible and strictly improves $\mathbf{c}\cdot\mathbf{z}^*$, which contradicts the definition of $\mathbf{z}^*$.

Define the event

$$\Omega_{k_0} \equiv \{\omega : D_i(\omega) \leq k_0,\ i = 1, .., m\}, \quad \text{where } k_0 = \frac{k\underline{a} - M}{\bar{a}m}.$$

We let k be sufficiently large so that $k_0 \geq 0$ (which is feasible because M does not depend on k). Since $y_j^* \geq -M$ $(1 \leq j \leq n)$, for all $\omega \in \Omega_{k_0}$,

$$\mathbf{A}_j \cdot \mathbf{D} \leq \frac{k\underline{a} - M}{\bar{a}m} \sum_{i=1}^m a_{ji} \leq k\underline{a} - M \leq y_j^* + k\underline{a}, \quad 1 \leq j \leq n.$$

Therefore the probability that (EC.6) does not hold satisfies

$$\mathbf{P}\{\exists\, j \in \{1, ..., n\} : \mathbf{A}_j \cdot \mathbf{D} > y_j^* + k\underline{a}\} \leq \mathbf{P}\{\omega \notin \Omega_{k_0}\}.$$

Since $z_i^* \geq -k$ $(1 \leq i \leq m)$ is feasible for (EC.4) on Ω_{k_0} and $\mathbf{z}^* \leq \mathbf{D}$,

$$G_1(\mathbf{y}^*, \boldsymbol{\alpha}^{(k)}) - G_2(\mathbf{y}^*) \geq -\mathbf{c} \cdot E[\mathbf{z}^* \mathbf{1}(\omega \notin \Omega_{k_0})] \geq -\sum_{i=1}^{m} c_i E[D_i \mathbf{1}(\omega \notin \Omega_{k_0})], \tag{EC.7}$$

where $\mathbf{1}()$ is the indicator function. Since for all $i = 1, .., m$,

$$E[D_i \mathbf{1}(\omega \notin \Omega_{k_0})] \leq k_0 \sum_{l=1}^{m} \mathbf{P}\{D_l \geq k_0\} + E[D_i \mathbf{1}(D_i \geq k_0)],$$

we can prove the right-hand side of (EC.7) converges to zero and thus (EC.5) is true by showing that

$$\lim_{k_0 \to \infty} E[D_i \mathbf{1}(D_i \geq k_0)] = 0, \quad 1 \leq i \leq m,$$

which holds for all $i = 1, ..., m$ because

$$E[D_i \mathbf{1}(D_i \geq k_0)] = \int_{k_0}^{\infty} \mathbf{P}\{D_i \geq x\} dx \leq E[D_i^2] \int_{k_0}^{\infty} x^{-2} dx = E[D_i^2]/k_0$$

and $E[D_i^2]$ $(1 \leq i \leq m$ $)$ is finite by assumption.

EC.2. Proof of Lemma 2

Let s_i^k be the k^{th} $(k = 1, 2, ...)$ realization of S_i and

$$s_i^{\max}(k) = \max\{s_i^1, s_i^2, ..., s_i^k\}.$$

We first prove that for all $k = 1, 2, ...,$

$$\frac{E[s_i^{\max}(k)]}{\sqrt{k}} \leq (1 + \eta_i) k^{-\frac{\delta}{2(2+\delta)}}. \tag{EC.8}$$

Let $\bar{F}_{i,d}(x)$ be the complimentary CDF of S_i $(1 \leq i \leq m)$. Then by Chebyshev's Inequality,

$$\begin{aligned}\mathbf{P}\{s_i^{\max}(k) > \sqrt{k}x\} &= 1 - \left[1 - \bar{F}_{i,d}(\sqrt{k}x)\right]^k \leq 1 - [1 - k\bar{F}_{i,d}(\sqrt{k}x)] \\ &\leq 1 - \left(1 - \frac{k\eta_i}{(\sqrt{k}x)^{2+\delta}}\right) = \eta_i k^{-\delta/2} x^{-(2+\delta)}.\end{aligned}$$

Let $\Delta = \delta/(2(2+\delta))$ and use the above

$$\begin{aligned}\frac{E[s_i^{\max}(k)]}{\sqrt{k}} &= \int_0^{\infty} \frac{\mathbf{P}\{s_i^{\max}(k) > x\} dx}{\sqrt{k}} = \int_0^{\infty} \mathbf{P}\{s_i^{\max}(k) > \sqrt{k}x\} dx \\ &\leq k^{-\Delta} + \int_{k^{-\Delta}}^{\infty} \eta_i k^{-\delta/2} x^{-(2+\delta)} dx \\ &< k^{-\Delta} + \eta_i k^{-\delta/2 + \Delta(1+\delta)} = (1 + \eta_i) k^{-\frac{\delta}{2(2+\delta)}}.\end{aligned}$$

To use (EC.8) to prove the lemma, let $\Lambda^{(L)}$ be the Possion random variable with mean $L\lambda$ $(1 \leq i \leq m)$,

$$\sup_{t-1 \leq \tau \leq t} \tilde{d}_i^{(L)}(\tau) \stackrel{d}{=} s_i^{\max}(\Lambda^{(L)}), \ 1 \leq i \leq m.$$

Let $p_k^{(L)} = \mathbf{P}\{\Lambda^{(L)} = k\}$. Then

$$E\left[\sup_{t-1\le\tau\le t} \tilde{d}_i^{(L)}(\tau)\right] = \sum_{k=0}^{\lfloor\lambda L\rfloor} p_k^{(L)} \frac{E[s_i^{\max}(k)]}{\sqrt{L}} + \sum_{k=\lfloor\lambda L\rfloor+1}^{\infty} p_k^{(L)} \frac{E[s_i^{\max}(k)]}{\sqrt{L}}. \tag{EC.9}$$

For all $k \le \lfloor\lambda L\rfloor$, $E[s_i^{\max}(k)] \le E[s_i^{\max}(\lfloor\lambda L\rfloor)]$. Thus

$$\sum_{k=0}^{\lfloor\lambda L\rfloor} p_k^{(L)} \frac{E[s_i^{\max}(k)]}{\sqrt{L}} \le \sqrt{\lambda}\frac{E[s_i^{\max}(\lfloor\lambda L\rfloor)]}{\sqrt{(\lfloor\lambda L\rfloor)}}.$$

By applying (EC.8) and observing that $\lambda L \le 2\lfloor\lambda L\rfloor$ $(1\le i\le m)$

$$\begin{aligned}\sqrt{\lambda}\frac{E[s_i^{\max}(\lfloor\lambda L\rfloor)]}{\sqrt{\lfloor\lambda L\rfloor}} &\le \sqrt{\lambda}(1+\eta_i)(\lfloor\lambda L\rfloor)^{-\frac{\delta}{2(2+\delta)}}\\ &\le \sqrt{\lambda}(1+\eta_i)(\lambda L/2)^{-\frac{\delta}{2(2+\delta)}}\\ &\le 2(1+\eta_i)\lambda^{\frac{1}{2+\delta}} L^{-\frac{\delta}{2(2+\delta)}}.\end{aligned} \tag{EC.10}$$

For all $k > \lfloor\lambda L\rfloor$,

$$\frac{p_k^{(L)}}{\sqrt{L}} = \frac{(\lambda L)^k e^{-\lambda L}}{k!\sqrt{L}} \le \sqrt{\frac{\lambda}{k}}\frac{(\lambda L)^{k-1}}{(k-1)!}e^{-\lambda L} = \sqrt{\frac{\lambda}{k}} p_{k-1}^{(L)}.$$

Using (EC.8) and observing that $k^{-\frac{\delta}{2(2+\delta)}} \le (\lambda L)^{-\frac{\delta}{2(2+\delta)}}$ when $k > \lfloor\lambda L\rfloor$,

$$\begin{aligned}\sum_{k=\lfloor\lambda L\rfloor+1}^{\infty} p_k^{(L)} \frac{E[s_i^{\max}(k)]}{\sqrt{L}} &\le \sqrt{\lambda}\sum_{k=\lfloor\lambda L\rfloor+1}^{\infty} p_{k-1}^{(L)} \frac{E[s_i^{\max}(k)]}{\sqrt{k}}\\ &\le (1+\eta_i)\sqrt{\lambda}\sum_{k=\lfloor\lambda L\rfloor+1}^{\infty} p_{k-1}^{(L)} k^{-\frac{\delta}{2(2+\delta)}}\\ &\le (1+\eta_i)\lambda^{\frac{1}{2+\delta}} L^{-\frac{\delta}{2(2+\delta)}}.\end{aligned} \tag{EC.11}$$

The lemma follows from (EC.9), (EC.10), and (EC.11).

EC.3. Proof of Lemma 3

Since $\hat{D}_i^{(L)}(0,\cdot)$ is a martingale (with respect to the filtration generated by $\{\mathcal{D}(t), t\ge 0\}$) and $|x|$ is a convex function, $|\hat{D}_i^{(L)}(0,\cdot)|$ is a sub-martingale $(1\le i\le m)$. We apply Doob's inequality to prove both results. For (34),

$$\begin{aligned}&E\left[\sup_{0\le\tau\le L^{-1/4}} |\hat{D}_i^{(L)}(0,\tau)|\right]\\ &= \int_0^{L^{-1/8}} \mathbf{P}\left\{\sup_{0\le\tau\le L^{-1/4}} |\hat{D}_i^{(L)}(0,\tau)| > x\right\} dx + \int_{L^{-1/8}}^{\infty} \mathbf{P}\left\{\sup_{0\le\tau\le L^{-1/4}} |\hat{D}_i^{(L)}(0,\tau)| > x\right\} dx\\ &\le L^{-1/8} + \int_{L^{-1/8}}^{\infty} E\left[(\hat{D}_i^{(L)}(0,L^{-1/4}))^2\right] x^{-2} dx\\ &= L^{-1/8}(1+\sigma_{ii}^2) \qquad\qquad 1\le i\le m.\end{aligned}$$

For (35),

$$\begin{aligned} E\left[\sup_{L^{-1/4}\le\tau\le 1}\left(|\hat{D}_i^{(L)}(0,\tau)|-\sqrt{L}\tau\kappa\right)^+\right] &\le E\left[\sup_{L^{-1/4}\le\tau\le 1}\left(|\hat{D}_i^{(L)}(0,\tau)|-L^{1/4}\kappa\right)^+\right] \\ &= \int_{L^{1/4}\kappa}^{\infty}\mathbf{P}\left\{\sup_{L^{-1/4}\le\tau\le 1}|\hat{D}_i^{(L)}(0,\tau)|>x\right\}dx \\ &\le \int_{L^{1/4}\kappa}^{\infty}E\left[(\hat{D}_i^{(L)}(0,1))^2\right]x^{-2}dx \\ &= \frac{\sigma_{ii}^2}{\kappa}L^{-1/4}, \quad 1\le i\le m. \end{aligned}$$

EC.4. Proof of Lemma 4

The proof is similar to the first part of the proof of Theorem 2. Following the same reasoning that led to (EC.3) and observing that $E[\hat{\mathbf{D}}^{(L)}]=\mathbf{0}$,

$$\hat{C}_+^{(L)}(\hat{\mathbf{y}})\ge -\zeta_j\hat{y}_j \quad \text{and} \quad \hat{C}_+^{(L)}(\hat{\mathbf{y}})\ge h_j\hat{y}_j, \quad 1\le j\le n.$$

By definition $\hat{\mathbf{D}}^{(L)}\ge -\sqrt{L}\boldsymbol{\mu}$, thus $\hat{\mathbf{y}}^{(L)}=\mathbf{0}$ and $\hat{\mathbf{z}}^{(L)}=\mathbf{0}\wedge\hat{\mathbf{D}}^{(L)}$ (where the minimum is taken component-wise) are feasible for (40), so that

$$\hat{C}_+^{(L)}(\hat{\mathbf{y}}^{(L)o})\le -\mathbf{c}\cdot E[\mathbf{0}\wedge\hat{\mathbf{D}}^{(L)}])\le \mathbf{c}\cdot E[|\hat{\mathbf{D}}^{(L)}|],$$

and thus

$$|\hat{y}_j^{(L)o}|\le\frac{\mathbf{c}\cdot E[|\hat{\mathbf{D}}^{(L)}|]}{h_j\wedge\zeta_j}, \quad 1\le j\le n,$$

where the right-hand-side is finite because

$$E[|\hat{D}_i^{(L)}|]\le 2+E[(\hat{D}_i^{(L)})^2]\le 2+\max\{\sigma_{11}^2,....,\sigma_{mm}^2\},$$

and σ_{ii}^2 $(1\le i\le m)$ are finite. Following the same reasoning, the above upper bound also applies to $\hat{y}_j^{(L)*}$ $(1\le j\le n)$, and thus the weighted average of $\hat{\mathbf{y}}^{(L)o}$ and $\hat{\mathbf{y}}^{(L)*}$, $\hat{\mathbf{y}}^{(L)\gamma}$.

EC.5. Proof of Lemma 5

For $t_2>t_1\ge 1$,

$$\begin{aligned} \tilde{\mathbf{B}}^{(L)}(t_2) &= \tilde{\mathbf{B}}^{(L)}(t_1)+\tilde{\mathbf{D}}^{(L)}(t_1,t_2)-\tilde{\mathbf{Z}}^{(L)}(t_1,t_2), \\ \tilde{\mathbf{I}}^{(L)}(t_2) &= \tilde{\mathbf{I}}^{(L)}(t_1)+\tilde{\mathbf{R}}^{(L)}(t_1-1,t_2-1)-A\tilde{\mathbf{Z}}^{(L)}(t_1,t_2). \end{aligned}$$

Under a base-stock policy,

$$\tilde{\mathbf{R}}^{(L)}(t_1-1,t_2-1)=A\tilde{\mathbf{D}}^{(L)}(t_1-1,t_2-1).$$

The above three equations imply that

$$\begin{aligned}\tilde{\mathbf{Q}}^{(L)}(t_2)-\tilde{\mathbf{Q}}^{(L)}(t_1) &= A\tilde{\mathbf{B}}^{(L)}(t_2)-\tilde{\mathbf{I}}^{(L)}(t_2)-(A\tilde{\mathbf{B}}^{(L)}(t_1)-\tilde{\mathbf{I}}^{(L)}(t_1))\\ &= A(\tilde{\mathbf{D}}^{(L)}(t_1,t_2)-\tilde{\mathbf{D}}^{(L)}(t_1-1,t_2-1))\\ &= A(\hat{\mathbf{D}}^{(L)}(t_1,t_2)-\hat{\mathbf{D}}^{(L)}(t_1-1,t_2-1)).\end{aligned}$$

Since $\tilde{\mathbf{B}}^{(L)*}(t)$ is Lipschitz continuous in $\tilde{\mathbf{Q}}^{(L)}(t)$, there exists ϑ such that for all $i=1,...,m$,

$$|\tilde{B}_i^{(L)*}(t_2)-\tilde{B}_i^{(L)*}(t_1)|\le\vartheta\sum_{j=1}^{n}|\tilde{Q}_j^{(L)}(t_2)-\tilde{Q}_j^{(L)}(t_1)|\le g\sum_{l=1}^{m}|\hat{D}_l^{(L)}(t_1,t_2)-\hat{D}_l^{(L)}(t_1-1,t_2-1)| \quad \text{(EC.12)}$$

where g is a constant, so (45) holds. The proof of (46) is similar to the above, using

$$\tilde{\mathbf{Q}}^{(L)}(t)-\tilde{\mathbf{Q}}^{(L)}(t^-)=A\tilde{\mathbf{d}}^{(L)}(t)-\tilde{\mathbf{r}}^{(L)}(t-1)=A(\tilde{\mathbf{d}}^{(L)}(t)-\tilde{\mathbf{d}}^{(L)}(t-1)).$$

EC.6. Proof of Theorem 3

The conclusion that

$$\lim_{L\to\infty}\underline{\hat{C}}^{(L)}=\hat{C}^*$$

follows from Theorem 2.2 in Robinson and Wets [26]. Specifically, $\hat{\mathbf{D}}^{(L)}$ $(L>0)$ weakly converges to $\boldsymbol{\xi}$. The recourse problem $\varphi(\hat{\mathbf{y}};\mathbf{D}^{\hat{(L)}})$ is always feasible and by Hoffman's Lemma, continuous in both $\hat{\mathbf{y}}$ and $\hat{\mathbf{D}}^{(L)}$. Since

$$\varphi(\hat{\mathbf{y}};\hat{\mathbf{D}}^{(L)})\le\mathbf{c}\cdot\hat{\mathbf{D}}^{(L)},$$

and $\hat{\mathbf{D}}^{(L)}$ $(L>0)$ have the same finite covariance matrix Σ, $\varphi(\hat{\mathbf{y}};\hat{\mathbf{D}}^{(L)})$ is uniformly integrable over $\{\hat{\mathbf{D}}^{(L)},L>0\}$. The function $\mathbf{h}\cdot\hat{\mathbf{y}}^{(L)}$ is continuous. By Lemma 4, the optimal solution $\hat{\mathbf{y}}^{(L)*}$ $(L>0)$ is bounded. Thus all conditions of Theorem 2.2 in Robinson and Wets (1987) are satisfied. Their conclusion (a) immediately leads to (47).

To prove

$$\lim_{L\to\infty}\hat{C}_+^{(L)*}=\lim_{L\to\infty}\underline{\hat{C}}^{(L)},$$

first observe that by Lemma 4, $\hat{\mathbf{y}}^{(L)*}$ is a feasible solution of the SP in (40) if L is sufficiently large. Hence it suffices to prove that

$$\lim_{L\to\infty}E[\hat{\varphi}(\hat{\mathbf{y}}^{(L)*};\hat{\mathbf{D}}^{(L)})]=\lim_{L\to\infty}E[\hat{\varphi}_+(\hat{\mathbf{y}}^{(L)*};\hat{\mathbf{D}}^{(L)})] \quad \text{(EC.13)}$$

Consider systems in which

$$\sqrt{L}\ge\frac{M}{\upsilon},\quad \text{where}\quad \upsilon=\underline{a}\min_{1\le i\le m}\{\mu_i\}.$$

Recall that $\underline{a}$ denotes the smallest non-zero element of A and $\bar{a}$ denotes the largest element. Define

$$\Omega_0^{(L)} = \left\{ \hat{D}_i^{(L)} \le -\frac{M}{\underline{a}} + \frac{\sqrt{L}\upsilon}{\bar{a}(m-1)}, \quad 1 \le i \le m \right\}.$$

On any sample path where $\omega \in \Omega_0^{(L)}$, the optimal solution of $\hat{\varphi}(\hat{\mathbf{y}}^{(L)*}, \hat{\mathbf{D}}^{(L)})$ must satisfy

$$\hat{z}_i^{(L)} \ge -\sqrt{L}\mu_i, \quad \text{for all} \quad i = 1, ..., m, \tag{EC.14}$$

which means that an optimal solution of $\hat{\varphi}(\hat{\mathbf{y}}^{(L)*}; \hat{\mathbf{D}}^{(L)})$ is a feasible solution of $\hat{\varphi}_+(\hat{\mathbf{y}}^{(L)*}; \hat{\mathbf{D}}^{(L)})$, so optimal values of the two LPs coincide.

To prove (EC.14), if $\hat{z}_i^{(L)} < -\sqrt{L}\mu_i$ for some i, then since $D_i^{(L)} \ge 0$.

$$\hat{D}_i^{(L)} = \frac{D_i^{(L)} - L\mu_i}{\sqrt{L}} \ge -\sqrt{L}\mu_i > \hat{z}_i^{(L)}.$$

For any j such that $a_{ji} > 0$, if $a_{jk} = 0$ for all $k \ne i$, then

$$a_{ji}\hat{z}_i^{(L)} < -a_{ji}\sqrt{L}\mu_i \le -\sqrt{L}\upsilon \le -M \le \hat{y}^{(L)*}.$$

Otherwise, if $a_{jk} > 0$ for some $k \ne i$, then

$$\begin{aligned}
\sum_{k=1}^{m} a_{jk}\hat{z}_k^{(L)} &= a_{ji}\hat{z}_i^{(L)} + \sum_{k \ne i} a_{jk}\hat{z}_k^{(L)} \\
&< -a_{ji}\sqrt{L}\mu_i + \sum_{k \ne i} a_{jk}\hat{D}_k^{(L)} \\
&\le -a_{ji}\sqrt{L}\mu_i - \left(\frac{M}{\underline{a}} - \frac{\sqrt{L}\upsilon}{\bar{a}(m-1)} \right) \sum_{k \ne i} a_{jk} \\
&\le -\sqrt{L}(a_{ji}\mu_i - \upsilon) - M \\
&\le \hat{y}_j^{(L)*}.
\end{aligned}$$

In either case, the value of $\hat{z}_i^{(L)}$ can be strictly increased to improve the objective.

On sample paths where $\omega \notin \Omega_0^{(L)}$, let

$$\hat{z}_i^{(L)} = \hat{D}_i^{(L)} \wedge (-M/\underline{a}), \quad 1 \le i \le m.$$

Because $\mathbf{D}^{(L)}$ is nonnegative and $\sqrt{L} \ge M/\upsilon$,

$$-\sqrt{L}\mu_i \le \hat{D}_i^{(L)} \text{ and } -\sqrt{L}\mu_i \le -\sqrt{L}\upsilon/\underline{a} \le -M/\underline{a}, \quad 1 \le i \le m,$$

which means that $-\sqrt{L}\mu_i \le \hat{z}_i^{(L)}$ $(1 \le i \le m)$. Moreover

$$\sum_{i=1}^{m} a_{ji}\hat{z}_i^{(L)} \le \sum_{i=1}^{m} a_{ji}(-M/\underline{a}) \le -M \le \hat{y}_j^{(L)*}, \quad 1 \le j \le n,$$

so $\hat{z}_i^{(L)}$ $(1 \le i \le m)$ is a feasible solution of $\hat{\varphi}_+(\hat{\mathbf{y}}^{(L)*}; \hat{\mathbf{D}}^{(L)})$.

Let $\bar{c} = \max\{c_1, ..., c_m\}$. Since $\varphi(\hat{\mathbf{y}}^{(L)*}; \hat{\mathbf{D}}^{(L)}) \le \mathbf{c} \cdot \hat{\mathbf{D}}^{(L)}$,

$$\begin{aligned} E[\varphi(\hat{\mathbf{y}}^{(L)*}, \hat{\mathbf{D}}^{(L)})] - E[\varphi_+(\hat{\mathbf{y}}^{(L)*}, \hat{\mathbf{D}}^{(L)})] &\le E\left[\sum_{i=1}^m c_i \left(\hat{D}_i^{(L)} - [\hat{D}_i^{(L)} \wedge (-M/\underline{a})]\right) \mathbf{1}(\omega \notin \Omega_0^{(L)})\right] \\ &\le \bar{c} \sum_{i=1}^m E\left[\left(|\hat{D}_i^{(L)}| + M/\underline{a}\right) \mathbf{1}\left(\hat{D}_i^{(L)} > -\frac{M}{\underline{a}} + \frac{\sqrt{L}\upsilon}{\bar{a}(m-1)}\right)\right], \end{aligned}$$

and (EC.13) follows because $E[|\hat{D}_i^{(L)}| + M/\underline{a}]$ is a finite and

$$\lim_{L\to\infty} \mathbf{P}\left(\hat{D}_i^{(L)} > -\frac{M}{\underline{a}} + \frac{\sqrt{L}\upsilon}{\bar{a}(m-1)}\right) = 0.$$

EC.7. Proof of Lemma 6

Let $\mathbf{y}^{(L)\gamma}$ be the base-stock levels in use. Define

$$\underline{C}^{(L)\gamma} = \mathbf{h} \cdot \mathbf{y}^{(L)\gamma} + \mathbf{b} \cdot E[\mathbf{D}^{(L)}] - E[\varphi(\mathbf{y}^{(L)\gamma}; \mathbf{D}^{(L)})].$$

Because the LP $\varphi(\mathbf{y}; \mathbf{D}^{(L)})$ is concave in $\mathbf{y}$,

$$\begin{aligned} E[\varphi(\mathbf{y}^{(L)\gamma}; \mathbf{D}^{(L)})] &\ge \gamma E[\varphi(\mathbf{y}^{(L)*}; \mathbf{D}^{(L)})] + (1-\gamma) E[\varphi(\mathbf{y}^{(L)o}; \mathbf{D}^{(L)})] \\ &\ge \gamma E[\varphi(\mathbf{y}^{(L)*}; \mathbf{D}^{(L)})] + (1-\gamma) E[\varphi_+(\mathbf{y}^{(L)o}; \mathbf{D}^{(L)})], \end{aligned}$$

where the second inequality holds because $\varphi(\mathbf{y}; \mathbf{D}^{(L)})$ relaxes the non-negativity constraints in $\varphi_+(\mathbf{y}; \mathbf{D}^{(L)})$. Therefore

$$\underline{C}^{(L)\gamma} \le \gamma \underline{C}^{(L)} + (1-\gamma) C_+^{(L)},$$

which, by Theorem 3, implies that

$$\lim_{L\to\infty} \frac{\underline{C}^{(L)\gamma}}{\sqrt{L}} \le \lim_{L\to\infty} \frac{\underline{C}^{(L)}}{\sqrt{L}}, \quad \gamma \in [0,1]. \tag{EC.15}$$

Applying (26) to the L^{th} system that uses $\mathbf{y}^{(L)\gamma}$ as base-stock levels,

$$\mathbf{I}^{(L)}(Lt) = \mathbf{y}^{(L)\gamma} + A\mathbf{B}^{(L)}(Lt) - A\mathbf{D}^{(L)}(Lt), \quad t \ge 1.$$

Scaling $\mathbf{I}^{(L)}(Lt)$ in the above and observing that $E[\mathbf{D}^{(L)}(Lt)] = L\boldsymbol{\mu}$ and $\mathbf{c} = \mathbf{b} + A\mathbf{h}$,

$$\frac{\mathbf{b} \cdot E[\mathbf{B}^{(L)}(Lt)] + \mathbf{h} \cdot E[\mathbf{I}^{(L)}(Lt)]}{\sqrt{L}} = \mathbf{c} \cdot E[\tilde{\mathbf{B}}^{(L)}(t)] + \mathbf{h} \cdot \hat{\mathbf{y}}^{(L)\gamma} \quad t \ge 1. \tag{EC.16}$$

Applying (27) in the L^{th} system yields

$$\mathbf{Q}^{(L)}(Lt) = A\mathbf{D}^{(L)}(Lt) - \mathbf{y}^{(L)\gamma} \stackrel{d}{=} A\mathbf{D}^{(L)} - \mathbf{y}^{(L)\gamma}.$$

Because of this equivalence in distribution, in the L^{th} system, (28) yields

$$\mathbf{c}\cdot E[\mathbf{B}^{(L)*}(t)] = \mathbf{c}\cdot E[\mathbf{D}^{(L)} - \mathbf{z}^{(L)*}], \tag{EC.17}$$

where given $\mathbf{y}^{(L)\gamma}$, $\mathbf{z}^{(L)*}$ maximizes the second LP in (28), transformed from the recourse problem of the lower bound SP (17). (For brevity, we suppress index γ in $\mathbf{z}^{(L)*}$ and $\mathbf{B}^{(L*)}(t)$.)

Applying (EC.16) to scaled inventory cost in the L^{th} system,

$$\frac{\mathcal{C}_L^\gamma}{\sqrt{L}} = \lim_{T\to\infty}\left\{\frac{1}{T}\int_1^{T+1}\frac{\mathbf{b}\cdot E[\mathbf{B}^{(L)}(Lt)] + \mathbf{h}\cdot E[\mathbf{I}^{(L)}(Lt)]}{\sqrt{L}}dt\right\} = \mathbf{h}\cdot\hat{\mathbf{y}}^{(L)\gamma} + \lim_{T\to\infty}\left\{\frac{1}{T}\int_1^{T+1}\mathbf{c}\cdot E[\tilde{\mathbf{B}}^{(L)}(t)]dt\right\}.$$

Following (EC.17) and (42),

$$\frac{\underline{C}^{(L)\gamma}}{\sqrt{L}} = \frac{\mathbf{h}\cdot E[\mathbf{y}^{(L)\gamma} - A\mathbf{z}^{(L)*}] + \mathbf{b}\cdot E[\mathbf{D}^{(L)} - \mathbf{z}^{(L)*}]}{\sqrt{L}} = \mathbf{h}\cdot\hat{\mathbf{y}}^{(L)\gamma} + \mathbf{c}\cdot E[\tilde{\mathbf{B}}^{(L)*}(t)].$$

Therefore

$$\frac{\mathcal{C}_L^\gamma - \underline{C}^{(L)\gamma}}{\sqrt{L}} = \lim_{T\to\infty}\left\{\frac{1}{T}\int_1^{T+1}\mathbf{c}\cdot\left(E[\tilde{\mathbf{B}}^{(L)}(t)] - E[\tilde{\mathbf{B}}^{(L)*}(t)]\right)dt\right\},$$

and the lemma follows from (EC.15) and that $\underline{C}^{(L)}$ is a lower bound.

References

Robinson S, Wets R (1987) Stability in two-stage stochastic programming. *SIAM J. Control and Optim.* 25(6):1409-1416.