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Asymptotically-optimal component allocation for Assemble-to-Order production–inventory systems

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ABSTRACT

We consider component allocation in Assemble-to-Order production–inventory systems. We prove that asymptotic optimality on the diffusion scale can be achieved under a continuous-review policy. We also show that in many systems, meeting this optimality criterion requires component reservation.

1. Introduction

We consider an Assemble-to-Order (ATO) production/inventory system that builds m products from n components. The Bill of Material (BOM) is given by a nonnegative integer matrix A, where a_{ji} is the amount of component j $(1 \le j \le n)$ needed to produce a unit of product i $(1 \le i \le m)$. Inventories are kept at the component level and the planning horizon starts from time 0. Let $\mathcal{D}_i(t)$ be the amount of demand for product i $(1 \le i \le m)$ arrived and $\mathcal{R}_j(t)$ be the amount of component j $(1 \le j \le n)$ produced during $[0, t]$. Denote

$$\mathcal{D}(t) = (\mathcal{D}_1(t), \dots, \mathcal{D}_m(t)) \quad \text{and}$$
$$\mathcal{R}(t) = (\mathcal{R}_1(t), \dots, \mathcal{R}_n(t)), \quad t \ge 0.$$

The problem is to define an allocation policy that decides which demands to serve at time t based on information available by that time, i.e., to determine

$$\mathcal{Z}(t) = (\mathcal{Z}_1(t), \dots, \mathcal{Z}_m(t)), \quad t \ge 0,$$

where $\mathcal{Z}_i(t)$ is the amount of demand i $(1 \le i \le m)$ served during $[0, t]$.

We consider a backlog model and denote backlog and inventory levels by

$$\mathbf{B}(t) = (B_1(t), \dots, B_m(t)) \quad \text{and}$$
$$\mathbf{I}(t) = (I_1(t), \dots, I_n(t)), \quad t \ge 0$$

respectively. Assume $\mathbf{B}(0) = \mathbf{I}(0) = \mathbf{0}$, so the allocation decision is subject to

$$\mathcal{Z}(t) \le \mathcal{D}(t) \quad \text{and} \quad A\mathcal{Z}(t) \le \mathcal{R}(t), \quad t \ge 0, \tag{1}$$

and backlog and inventory levels are determined by

$$\mathbf{B}(t) = \mathcal{D}(t) - \mathcal{Z}(t), \quad \text{and} \quad \mathbf{I}(t) = \mathcal{R}(t) - A\mathcal{Z}(t), \quad t \ge 0. \tag{2}$$

Let $\delta > 0$ be the discount rate, p_i be the price of product i $(1 \le i \le m)$, b_i be the per-unit backlog cost of product i $(1 \le i \le m)$, and h_j be the per-unit inventory holding cost of component j $(1 \le j \le n)$. Our objective is to maximize the following Net Present Value (NPV) of the expected profit

$$E\left[\sum_{i=1}^{m}\int_0^\infty e^{-\delta t} p_i d\mathcal{Z}_i(t) - \int_0^\infty e^{-\delta t}\left(\sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t)\right) dt\right]. \tag{3}$$

The problem is known to be difficult and optimal policies have only been developed for special systems with memoryless demand and production processes [1,5]. For systems in the "high-volume" asymptotic regime, Plambeck and Ward develop pricing, capacity, and allocation policies that are asymptotically optimal on the diffusion scale [6]. Under their approach, the percentage difference of the NPV from its optimum converges to zero as demand arrival rates increase. Most relevant to our work is their allocation policy, which is a periodic-review policy with the review interval inversely related to a norm of demand arrival rates. By

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this design, sufficient safety stocks of demands and components are accumulated in each period to offset impacts of randomness. This policy has been applied to ATO inventory systems with N and W structures [4]. The latter work makes component ordering and allocation decisions dynamically over time for asymptotically optimal inventory management.

While asymptotically optimal, the above allocation policy does not allow managers to act immediately on continuously-arriving demands. Outside the high-volume regime, demands arrive at slow rates, and thus can lead to an excessively long review interval and high inventory costs. We address this limitation by proving that the same level of asymptotic optimality can be attained under continuous-review allocations. Our candidate policies are those defined by the Allocation Principle in [7], which have been proved to be asymptotically optimal for ATO inventory systems (components are ordered from uncapacitated suppliers instead of produced under capacity constraints) with identical lead times, and shown to outperform alternative approaches outside the asymptotic regime [2,3,7]. While applying these policies to our production/inventory systems is straightforward, proving they remain asymptotically optimal requires a new analysis. Unlike [7], which features a stationary system and a long-run average cost objective function, our systems are non-recurrent and our objective is a discounted value function. Developing a proof of asymptotic optimality, under a relaxed moment condition from [6], is a major contribution of this paper.

Following the Allocation Principle in [7] often requires reserving components for future high-value demands [3,7]. Reservation, which holds back components from some existing demands, is an insurance against hindsight regrets about current allocation decisions. However, in the high-volume asymptotic regime, these regrets can be addressed rapidly by fast production of component replenishments. Thus one may ask whether reservation is necessary for asymptotic optimality. To answer this question, we prove that in many systems, any policy that does not reserve components cannot be asymptotically optimal.

We will present problem background in Section 2, prove asymptotic optimality in Section 3, and demonstrate the need for reservation in Section 4.

2. Preliminaries

By applying the equality

$$\int_0^\infty e^{-\delta t} p_i d\mathcal{Z}_i(t) = \int_0^\infty e^{-\delta t}\delta p_i \mathcal{Z}_i(t)dt, \quad 1 \le i \le m,$$

and using (2) to replace $\mathcal{Z}(t)$ and $\mathbf{I}(t)$, we rewrite the objective function (3) as

$$E\left[\int_0^\infty e^{-\delta t}\left(\sum_{i=1}^m \left[\delta p_i + \sum_{j=1}^n h_j a_{ji}\right]\mathcal{D}_i(t) - \sum_{i=1}^m c_i B_i(t) - \sum_{j=1}^n h_j \mathcal{R}_j(t)\right)dt\right],$$

where

$$c_i = \delta p_i + b_i + \sum_{j=1}^n a_{ji} h_j, \quad 1 \le i \le m$$

is the total value of serving a unit of demand for product i $(1 \le i \le m)$. Given $\mathcal{D}(t)$ and $\mathcal{R}(t)$, maximizing (3) is equivalent to choosing $\mathbf{B}(t)$ $(t \ge 0)$ to minimize

$$\mathcal{C} = E\left[\int_0^\infty e^{-\delta t}\sum_{i=1}^m c_i B_i(t)dt\right]. \tag{4}$$

Using (2), we define

$$\mathbf{Q}(t) \equiv A\mathcal{D}(t) - \mathcal{R}(t) = A\mathbf{B}(t) - \mathbf{I}(t), \tag{5}$$

as component shortage (for all j such that $Q_j(t) > 0$) or surplus (for all j such that $Q_j \le 0$) at time t $(t \ge 0)$. We then rewrite constraints in (1) as

$$\mathbf{B}(t) \ge 0 \quad \text{and} \quad A\mathbf{B}(t) \ge \mathbf{Q}(t), \quad t \ge 0.$$

Let $\mathbf{B}^*(t)$ $(t \ge 0)$ be the optimal solution of the Linear Program (LP)

$$\min_{\mathbf{B}\ge\mathbf{0}}\left\{\sum_{i=1}^m c_i B_i \middle| A\mathbf{B} \ge \mathbf{Q}(t)\right\}. \tag{6}$$

Following the discussion in Section 3.2 of [7], we can let $\mathbf{B}^*(t)$ to be Lipschitz-continuous in $\mathbf{Q}(t)$. Observe that the objective value $\mathcal{C}$ in (4) has a lower bound

$$\underline{\mathcal{C}} = E\left[\int_0^\infty e^{-\delta t}\sum_{i=1}^m c_i \mathbf{B}^*(t)dt\right]. \tag{7}$$

We refer to $\mathbf{B}^*(t)$ $(t \ge 0)$ as backlog targets. While reaching these targets at all times is generally impossible, the LP (6) that defines them is closely related to the aforementioned feasible policies. Component allocation in [6] relies on solving an LP that is equivalent to (6) with an additional constraint:

$$\min_{\mathbf{B}\ge\mathbf{0}}\left\{\sum_{i=1}^m c_i B_i \middle| A\mathbf{B} \ge \mathbf{Q}(t), \mathbf{B} \le \mathbf{B}^-(t)\right\}, \tag{8}$$

where $\mathbf{B}^-(t)$ denotes backlog levels at t $(t \ge 0)$ before serving demands at that time. Demands are served to reduce their backlog levels to $\mathbf{B}^0(t)$, the optimal solution of (8), which are feasible to reach because $\mathbf{B}^0(t) \le \mathbf{B}^-(t)$. However, reaching $\mathbf{B}^0(t)$ $(t \ge 0)$ does not optimize (4). Moreover, to be asymptotically optimal, components can only be allocated periodically with a carefully-chosen period length [6].

The Allocation Principle in [7] uses $\mathbf{B}^*(t)$ directly and requires that

1. Backlog $B_i(t)$ $(1 \le i \le m)$ should not exceed its target $B_i^*(t)$ when all components needed to reduce the excess are available, i.e.,

$$[B_i(t) - B_i^*(t)]^+ \wedge [\min_{j:a_{ji}>0}\{(I_j(t) - a_{ji} + 1)^+\}] = 0, \quad 1 \le i \le m, \ t \ge 0. \tag{9}$$

2. Backlog $B_i(t)$ $(1 \le i \le m)$ should not be reduced when it is below or at the target $B_i^*(t)$.

These two requirements specify to different allocation policies in different systems [2,3]. Demands are served continuously over time, but components can be held back from some low-value demands [3,7].

3. Asymptotic analysis

The same as in [6], we assume demand arrival and component production follow independent renewal processes. Define $m + n$ sequences of i.i.d. random variables, where x_i^l $(l = 1, \ldots)$ are inter-arrival times of demand i $(1 \le i \le m)$ and y_j^l $(l = 1, \ldots,)$ are production times of component j $(1 \le j \le n)$. Assume x_i^1 $(1 \le i \le m)$ and y_j^1 $(1 \le j \le n)$ have unit means and finite second moments, which relaxes the assumption in [6] that requires finite $(2 + \epsilon)$ moments $(\epsilon > 0)$.

Consider a series of systems $k = 1, \ldots,$ with $k\lambda_i$ as the arrival rate of demand i $(1 \le i \le m)$ and $k\mu_j$ as the production rate of component j ($1 \leq j \leq n$). Define demand and production processes respectively by

$$\mathcal{D}_i^{(k)}(t) \equiv \max\left\{L : \sum_{l=1}^{L} x_i^l \leq k\lambda_i t\right\}, \quad 1 \leq i \leq m,\ t \geq 0,$$

and $$\mathcal{R}_j^{(k)}(t) \equiv \max\left\{L : \sum_{l=1}^{L} y_j^l \leq k\mu_j t\right\}, \quad 1 \leq j \leq n,\ t \geq 0.$$

Denote the corresponding amounts occurred during a period $[t_1, t_2]$ by

$$\mathbf{D}^{(k)}(t_1, t_2) \equiv \mathcal{D}^{(k)}(t_2) - \mathcal{D}^{(k)}(t_1) \quad \text{and}$$
$$\mathbf{R}^{(k)}(t_1, t_2) \equiv \mathcal{R}^{(k)}(t_2) - \mathcal{R}^{(k)}(t_1),$$

respectively ($0 \leq t_1 < t_2$). In [6], demand and production rates are allowed to have a small imbalance, $\theta_j^{(k)}$ ($1 \leq j \leq n$), that satisfies

$$\lim_{k\to\infty} \sqrt{k}\theta_j^{(k)} \equiv \lim_{k\to\infty} \sqrt{k}\left(\sum_{i=1}^{m} a_{ji}\lambda_i^{(k)} - \mu_j^{(k)}\right) = \text{constants}. \tag{10}$$

Our results apply to this assumption, which is evident from their proofs: under (10), $\theta_j^{(k)}$ ($1 \leq j \leq n$) would have been dominated by other terms. Nevertheless, for brevity of presentation, we impose a slightly stronger assumption that

$$\sum_{i=1}^{m} a_{ji}\lambda_i - \mu_j = 0, \quad 1 \leq j \leq n.$$

For systems $k = 1, \ldots$, we center and scale demand and production processes by

$$\hat{\mathcal{D}}_i^{(k)}(t) \equiv \left(\mathcal{D}_i^{(k)}(t) - k\lambda_i t\right)/\sqrt{k}, \quad 1 \leq i \leq m,\ t \geq 0,$$
$$\hat{\mathcal{R}}_j^{(k)}(t) \equiv \left(\mathcal{R}_j^{(k)}(t) - k\mu_j t\right)/\sqrt{k}, \quad 1 \leq j \leq n,\ t \geq 0.$$

For a period $[t_1, t_2]$ where $0 \leq t_1 < t_2$, define

$$\hat{D}_i^{(k)}(t_1, t_2) \equiv \left(D_i^{(k)}(t_1, t_2) - k\lambda_i(t_2 - t_1)\right)/\sqrt{k}, \quad 1 \leq i \leq m,$$
$$\hat{R}_j^{(k)}(t_1, t_2) \equiv \left(R_j^{(k)}(t_1, t_2) - k\mu_j(t_2 - t_1)\right)/\sqrt{k}, \quad 1 \leq j \leq n. \tag{11}$$

For $t \geq 0$, we scale backlogs, inventories, and component shortage/surplus by

$$\tilde{\mathbf{B}}^{(\mathbf{k})}(t) = \mathbf{B}^{(k)}(t)/\sqrt{k}, \qquad \tilde{\mathbf{I}}^{(\mathbf{k})}(t) = \mathbf{I}^{(k)}(t)/\sqrt{k}, \quad \text{and}$$
$$\tilde{\mathbf{Q}}^{(\mathbf{k})}(t) = \mathbf{Q}^{(k)}(t)/\sqrt{k}.$$

Our analysis will use two properties of these processes.

First, let $\mathcal{B}(t)$ ($t \geq 0$) be a standard Brownian Motion. By FCLT,

$$\hat{\mathcal{D}}_i^{(k)}(t) \xrightarrow{d} \sigma_i\mathcal{B}(t) \quad (1 \leq i \leq m),\ t \geq 0,$$
$$\hat{\mathcal{R}}_j^{(k)}(t) \xrightarrow{d} \gamma_j\mathcal{B}(t) \quad (1 \leq j \leq n),\ t \geq 0, \tag{12}$$

where σ_i ($1 \leq i \leq m$) and γ_j ($1 \leq j \leq n$) are constants.

Second, following the proof of Lemma 2 in [6], which does not require finite $2+\epsilon$ ($\epsilon > 0$) moments, there exist constants κ_1 and κ_2 such that

$$E[\sup_{0\leq s\leq t} |\hat{\mathcal{R}}_j^{(k)}(s)|^2] \leq \kappa_1 t + \kappa_2 \quad (1 \leq j \leq n, t \geq 0),$$

and $$E[\sup_{0\leq s\leq t} |\hat{\mathcal{D}}_i^{(k)}(s)|^2] \leq \kappa_1 t + \kappa_2 \quad (1 \leq i \leq m, t \geq 0). \tag{13}$$

Let $\mathbf{B}^{(k)*}(t)$, $\mathcal{C}^{(k)}$, and $\underline{\mathcal{C}}^{(k)}$ be corresponding variables of $\mathbf{B}^*(t)$, $\mathcal{C}$, and $\underline{\mathcal{C}}$ in system k ($k = 1, \ldots$) respectively. By (4) and (7), if under an allocation policy,

$$\lim_{k\to\infty} E\left[\int_0^\infty e^{-\delta t}\left(B_i^{(k)}(t) - B_i^{(k)*}(t)\right)dt\right]/\sqrt{k} = 0,$$
$$1 \leq i \leq m, \tag{14}$$

then the objective value converges to its lower bound on the diffusion scale, i.e.,

$$\lim_{k\to\infty}\left(\mathcal{C}^{(k)} - \underline{\mathcal{C}}^{(k)}\right)/\sqrt{k} = 0. \tag{15}$$

FCLT implies that $\underline{\mathcal{C}}^{(k)}$ is on the order of $\sqrt{k}$. Therefore, when (15) holds, the percentage difference of the objective value from its optimum diminishes to zero.

Condition (14) is satisfied by the periodic-review policy in [6], which sets the review interval on the order of $k^{-2/3}$ to make the impact of the additional constraint in (8), $\mathbf{B} \leq \mathbf{B}^-(t)$, invisible on the diffusion scale. To prove (14) can be attained under any continuous-review policy that satisfies the Allocation Principle in [7], we first extract two properties of the principle from the latter paper and present them as Lemmas 1 and 2. Lemma 1 implies that when the backlog level of a product significantly exceeds its target, another product must have its backlog level below the target. Lemma 2 bounds the shortfall of a product's backlog from its target. Our main theorem, Theorem 1, shows that this bound is on the order of $o(\sqrt{k})$, so any difference between a product's backlog level and its target disappears on the diffusion scale.

Lemma 1. *Let $\underline{a}$ be the smallest non-zero element and $\bar{a}$ be the largest element of matrix A. Define $\phi = 1 - 1/\bar{a}$. Then under any policy that satisfies the aforementioned Allocation Principle,*

$$(B_i(t) - B_i^*(t))^+ \leq \phi + \frac{\bar{a}}{\underline{a}}\sum_{l=1}^{m}(B_l^*(t) - B_l(t))^+,$$
$$1 \leq i \leq m, t \geq 0. \tag{16}$$

Proof. From (9), if $B_i(t) > B_i^*(t)$, then there exists component j' such that $a_{j'i} > 0$ and $I_{j'}(t) \leq a_{j'i} - 1$. Apply the constraint in (6) to component j',

$$\sum_{l=1}^{m} a_{j'l}B_l^*(t) \geq Q_{j'}(t) = \sum_{l=1}^{m} a_{j'l}B_l(t) - I_{j'}(t)$$
$$\geq \sum_{l=1}^{m} a_{j'l}B_l(t) - (a_{j'i} - 1).$$

Since $a_{j'i} - 1 \leq a_{j'i}\phi$, the above implies that

$$\phi \geq \sum_{l=1}^{m}\frac{a_{j'l}}{a_{j'i}}(B_l(t) - B_l^*(t))$$
$$= \sum_{l=1}^{m}\frac{a_{j'l}}{a_{j'i}}[(B_l(t) - B_l^*(t))^+ - (B_l^*(t) - B_l(t))^+]$$
$$\geq (B_i(t) - B_i^*(t))^+ - \frac{\bar{a}}{\underline{a}}\sum_{l=1}^{m}(B_l^*(t) - B_l(t))^+,$$

and (16) follows by rearranging terms. ■

Lemma 2. *For any $t > 0$ and demand i such that $B_i^*(t) > B_i(t)$, define*

$$t_i = \sup\{\tau : 0 \leq \tau \leq t \text{ and } B_i(\tau) > B_i^*(\tau)\}. \tag{17}$$

Then there exist constants $h_1 > 0$ *and* $h_2 > 0$ *such that*

$$B_i^*(t) - B_i(t) \le h_1 \sum_{j=1}^{n} |Q_j(t) - Q_j(t_i)| + h_2 \sum_{j=1}^{n} |Q_j(t_i) - Q_j(t_i^-)| - D_i(t_i, t). \tag{18}$$

Proof. By the definition of t_i, $B_i(t) \le B_i^*(t)$ during $[t_i, t]$. Under the Allocation Principle, no demand i is served during that period, and thus

$$B_i(t) = B_i(t_i) + D_i(t_i, t). \tag{19}$$

Demand i is served at t_i only if $B_i^-(t_i) > B_i^*(t_i)$, so (17) implies that

$$B_i(t_i) = B_i^-(t_i) \wedge B_i^*(t_i) = B_i^*(t_i) - (B_i^*(t_i) - B_i^-(t_i))^+.$$

By the definition of t_i, $B_i(t_i^-) > B_i^*(t_i^-)$. Because $B_i^-(t_i)$ precedes allocation at t_i, $B_i^-(t_i) \ge B_i(t_i^-) > B_i^*(t_i^-)$. Thus the above inequalities imply that

$$B_i(t_i) \ge B_i^*(t_i) - |B_i^*(t_i) - B_i^*(t_i^-)|. \tag{20}$$

Eqs. (19) and (20) imply that

$$B_i^*(t) - B_i(t) \le B_i^*(t) - B_i^*(t_i) + |B_i^*(t_i) - B_i^*(t_i^-)| - D_i(t_i, t),$$

and (18) follows because $B_i^*(t)$ is Lipschitz continuous in $\mathbf{Q}(t)$. ■

Theorem 1. *Under any allocation policy that satisfies the Allocation Principle,*

$$\lim_{k\to\infty} E\left[\int_0^\infty e^{-\delta t} \sum_{i=1}^{m} c_i \tilde{B}_i^{(k)}(t)dt\right] = \lim_{k\to\infty} E\left[\int_0^\infty e^{-\delta t} \sum_{i=1}^{m} c_i \tilde{B}_i^{(k)*}(t)dt\right].$$

Hence the policy is asymptotically optimal on the diffusion scale.

Proof. Apply Lemma 1 to system k ($k = 1, \ldots$),

$$E\left[\int_0^\infty e^{-\delta t}(\tilde{B}_i^{(k)}(t) - \tilde{B}_i^{(k)*}(t))dt\right] \le \frac{\phi}{\sqrt{k}} \int_0^\infty e^{-\delta t}dt + \frac{\bar{a}}{\underline{a}} \times E\left[\int_0^\infty e^{-\delta t} \sum_{l=1}^{m} (\tilde{B}_l^{(k)*}(t) - \tilde{B}_l^{(k)}(t))^+ dt\right], \quad 1 \le i \le m.$$

Since δ and ϕ are constants, the above implies that (14) holds if

$$\lim_{k\to\infty} E\left[\int_0^\infty e^{-\delta t}(\tilde{B}_i^{(k)*}(t) - \tilde{B}_i^{(k)}(t))^+ dt\right] = 0, \quad 1 \le i \le m. \tag{21}$$

Let A_j be the jth row of matrix A ($1 \le j \le n$). Define

$$\eta_j = \sum_{i=1}^{m} a_{ji} + 1 \quad (1 \le j \le n) \quad \text{and}$$

$$\hat{a}_l = \sum_{j=1}^{n} a_{jl} \quad (1 \le l \le m).$$

Let $t_i^{(k)}$ be the time as defined in (17) for system k ($k = 1, 2 \ldots,$). Since $\mathcal{D}^{(k)}(t)$ and $\mathcal{R}^{(k)}(t)$ are renewal processes with step size 1,

$$|Q_j^{(k)}(t_i^{(k)}) - Q_j^{(k)}(t_i^{(k)-})| = |A_j \mathbf{D}^{(k)}(t_i^{(k)-}, t_i^{(k)}) - R_j^{(k)}(t_i^{(k)-}, t_i^{(k)})| \le \eta_j,$$

for all $j = 1, \ldots, n$. Apply the above and (5) to substitute Qs in (18),

$$B_i^{(k)*}(t) - B_i^{(k)}(t) \le G^{(k)}(t), \quad t \ge 0, \tag{22}$$

where

$$G^{(k)}(t) \equiv h_1 \sum_{j=1}^{n} \left|A_j \mathbf{D}^{(k)}(t_i^{(k)}, t) - R_j^{(k)}(t_i^{(k)}, t)\right| + h_2 \sum_{j=1}^{n} \eta_j - D_i^{(k)}(t_i^{(k)}, t).$$

Use (11) to center and scale $\mathbf{D}^{(k)}(t_i^{(k)}, t)$ and $\mathbf{R}^{(k)}(t_i^{(k)}, t)$. Define

$$\tilde{Y}^{(k)}(t) \equiv h_1 \sum_{l=1}^{m} |\hat{a}_l \hat{D}_l^{(k)}(t_i^{(k)}, t)| + h_1 \sum_{j=1}^{n} |\hat{R}_j^{(k)}(t_i^{(k)}, t)| - (t - t_i^{(k)})\sqrt{k}\lambda_i - \hat{D}_i^{(k)}(t_i^{(k)}, t), \quad t \ge 0.$$

Then

$$\tilde{Y}^{(k)}(t) \ge \frac{G^{(k)}(t) - h_2 \sum_{j=1}^{n} \eta_j}{\sqrt{k}}, \quad t \ge 0.$$

Because η_j ($1 \le j \le n$) are constants, (22) implies that (21) holds if

$$\lim_{k\to\infty} E\left[\int_0^\infty e^{-\delta t}(\tilde{Y}^{(k)}(t))^+ dt\right] = 0, \tag{23}$$

which we prove next. Define

$$\varphi \equiv \lambda_i \Big/ \left(h_1 \sum_{l=1}^{m} \hat{a}_l + nh_1 + 1\right) > 0.$$

Then

$$(\tilde{Y}^{(k)}(t))^+ \le h_1 \sum_{l=1}^{m} \hat{a}_l \left(|\hat{D}_l^{(k)}(t_i^{(k)}, t)| - (t - t_i^{(k)})\sqrt{k}\varphi\right)^+ + \left(|\hat{D}_i^{(k)}(t_i^{(k)}, t)| - (t - t_i^{(k)})\sqrt{k}\varphi\right)^+ + h_1 \sum_{j=1}^{n} \left(|\hat{R}^{(k)}(t_i^{(k)}, t)| - (t - t_i^{(k)})\sqrt{k}\varphi\right)^+,$$

which allows us to prove (23) by showing that

$$\lim_{k\to\infty} E\left[\int_0^\infty e^{-\delta t} \sup_{0\le s\le t} \left(|\hat{D}_l^{(k)}(s, t)| - (t - s)\sqrt{k}\varphi\right)^+ dt\right] = 0, \quad 1 \le l \le m,$$

and $\lim_{k\to\infty} E\left[\int_0^\infty e^{-\delta t} \sup_{0\le s\le t} \left(|\hat{R}_j^{(k)}(s, t)| - (t - s)\sqrt{k}\varphi\right)^+ dt\right] = 0, \quad 1 \le j \le n.$

Below we prove the first equation. A similar proof applies to the second one.

For any given l, define

$$\Psi_1^{(k)} \equiv E\left[\int_0^\infty e^{-\delta t} \sup_{0\le s\le t} |\hat{D}_l^{(k)}(s, t)| \mathbf{1}(t - s \le k^{-1/4})dt\right]$$

$$\Psi_2^{(k)} \equiv E\left[\int_0^\infty e^{-\delta t}(\sup_{0\le s\le t} |\hat{D}_l^{(k)}(s, t)| - (t - s)\sqrt{k}\varphi)^+ \mathbf{1}(t - s > k^{-1/4})dt\right].$$

Then

$$E\left[\int_0^\infty e^{-\delta t}\sup_{0\le s\le t}\left(|\hat{D}_l^{(k)}(s,t)|-(t-s)\sqrt{k}\varphi\right)^+dt\right]$$
$$\le \Psi_1^{(k)}+\Psi_2^{(k)},$$

so we only need to prove that $\Psi_i^{(k)}\to 0$ as $k\to\infty$ $(i=1,2)$.

By Tonelli's Theorem,

$$\Psi_1^{(k)}=\int_0^\infty e^{-\delta t}E\left[\sup_{(t-k^{-1/4})^+\le s\le t}|\hat{D}_l^{(k)}(s,t)|\right]dt.$$

For $k=1,\ldots,$ and $t\ge 0$,

$$\sup_{(t-k^{-1/4})^+\le s\le t}|\hat{D}_l^{(k)}(s,t)|\le 2\sup_{0\le s\le t}|\hat{\mathcal{D}}_l^{(k)}(s)|$$
$$\le 1+\sup_{0\le s\le t}|\hat{\mathcal{D}}_l^{(k)}(s)|^2, \tag{24}$$

and by (13),

$$E\left[(1+\sup_{0\le s\le t}|\hat{\mathcal{D}}_l^{(k)}(s)|^2)e^{-\delta t}\right]\le e^{-\delta t}(1+\kappa_1 t+\kappa_2),$$

where the right-hand side is integrable over $[0,\infty)$. Thus if for each $t>0$,

$$\lim_{k\to\infty}E\left[e^{-\delta t}\sup_{(t-k^{-1/4})^+\le s\le t}|\hat{D}_l^{(k)}(s,t)|\right]=0, \tag{25}$$

then we can apply the Dominated Convergence Theorem to exchange the limit and integral signs to arrive at

$$\lim_{k\to\infty}\Psi_1^{(k)}=\int_0^\infty e^{-\delta t}\lim_{k\to\infty}E\left[\sup_{(t-k^{-1/4})^+\le s\le t}|\hat{D}_l^{(k)}(s,t)|\right]dt=0.$$

To prove (25), for any $\epsilon>0$, choose $k_0\ge 1/t^4$ (so $t-k_0^{-1/4}\ge 0$) such that

$$e^{-\delta t}E[\sup_{0\le s\le k_0^{-1/4}}|\sigma_l\mathcal{B}(s)|]<\epsilon/2, \tag{26}$$

($\mathcal{B}(s)$ is a Standard Brownian Motion). By (12) and the Continuous Mapping Theorem,

$$\sup_{t-k_0^{-1/4}\le s\le t}|\hat{D}_l^{(k)}(s,t)|\xrightarrow{d}\sup_{0\le s\le k_0^{-1/4}}|\sigma_l\mathcal{B}(s)|\quad\text{as } k\to\infty,$$

for any given k_0. For each t, (13) and (24) imply that

$$\{e^{-\delta t}\sup_{t-k_0^{-1/4}\le s\le t}|\hat{D}_l^{(k)}(s,t)|\},\quad k=1,2,\ldots,$$

is uniformly integrable. So for given ϵ and k_0, $\exists k_1$ such that for all $k\ge k_1$,

$$e^{-\delta t}E[\sup_{t-k_0^{-1/4}\le s\le t}|\hat{D}_l^{(k)}(s,t)|]\le e^{-\delta t}E[\sup_{0\le s\le k_0^{-1/4}}|\sigma_l\mathcal{B}(s)|]$$
$$+\epsilon/2, \tag{27}$$

and (25) follows from (26) and (27) because for all $k\ge\max(k_0,k_1)$,

$$e^{-\delta t}E[\sup_{(t-k^{-1/4})^+\le s\le t}|\hat{D}_l^{(k)}(s,t)|]\le e^{-\delta t}E[\sup_{t-k_0^{-1/4}\le s\le t}|\hat{D}_l^{(k)}(s,t)|]$$
$$<\epsilon.$$

To prove $\lim_{k\to\infty}\Psi_2^{(k)}=0$, replace $(t-s)$ with $k^{-1/4}$ in its definition,

$$\Psi_2^{(k)}\le\int_0^\infty e^{-\delta t}E[(\sup_{0\le s\le t}|\hat{D}_l^{(k)}(s,t)|-k^{1/4}\varphi)^+]dt. \tag{28}$$

Since $\hat{D}_l^{(k)}(s,t)=\hat{\mathcal{D}}_l^{(k)}(t)-\hat{\mathcal{D}}_l^{(k)}(s)$ and because of (13),

$$E[(\sup_{0\le s\le t}|\hat{D}_l^{(k)}(s,t)|-k^{1/4}\varphi)^+]$$
$$\le E[(\sup_{0\le s\le t}|2\hat{\mathcal{D}}_l^{(k)}(s)|-k^{1/4}\varphi)^+]$$
$$=2\int_{k^{1/4}\varphi/2}^\infty\mathbb{P}\left(\sup_{0\le s\le t}|\hat{\mathcal{D}}_l^{(k)}(s)|\ge x\right)dx$$
$$\le 2\int_{k^{1/4}\varphi/2}^\infty\frac{E[\sup_{0\le s\le t}|\hat{\mathcal{D}}_l^{(k)}(s)|^2]}{x^2}dx$$
$$\le 2\int_{k^{1/4}\varphi/2}^\infty\frac{\kappa_1 t+\kappa_2}{x^2}dx.$$
$$=2(k^{1/4}\varphi/2)^{-1}(\kappa_1 t+\kappa_2),\quad t\ge 0.$$

Applying the above to (28) proves the result. ■

4. Need for component reservation

We consider systems with the BOM

$$A=\begin{pmatrix}1 & 1 & 0 & 0\ldots0\\ 1 & 0 & 1 & 0\ldots0\\ & & \tilde{A} & \end{pmatrix}. \tag{29}$$

Slightly deviating from the general notation, we index products by $0,1,\ldots,m-1$. The first two rows of A show that component i $(i=1,2)$ is used by products 0 and i by one unit each. The usage of other components is given by the rest of the matrix. For simplicity, we assume that all entries in A are binary. So any policy that does not reserve component satisfies the condition that

$$B_i(t)\wedge\prod_{j:a_{ji}=1}I_j(t)=0,\quad 0\le i\le m-1,\ t\ge 0, \tag{30}$$

i.e., no demand can have backlog if components to serve it are all available.

Theorem 2. *For systems in which the BOM is given by (29); values of serving products* 0, 1, *and* 2 *satisfy* $c_0>c_1+c_2$; *and demand arrival and component production follow* $m+n$ *independent Poisson processes, any policy that satisfies (30) is not asymptotically optimal, i.e., under such a policy,*

$$\lim_{k\to\infty}\left(\mathcal{C}^{(k)}-\underline{\mathcal{C}}^{(k)}\right)/\sqrt{k}>0. \tag{31}$$

We prove the theorem by showing that during a period $[1/2,1]$, there is a set of sample paths $(\Omega^{(k)})$ associated with a strictly positive probability (Eq. (35)). On these paths, the difference between the discounted inventory cost and its minimum is on the order of $\sqrt{k}$ (Eq. (36)).

Proof. Let $\Delta>0$ be a constant. Define the following sets of sample paths

$$\Omega^{(k)}=\mathsf{E}_1^{(k)}\wedge\mathsf{E}_2^{(k)}\wedge\mathsf{E}_3^{(k)}\wedge\mathsf{E}_4^{(k)}\wedge\mathsf{E}_5^{(k)}\wedge\mathsf{E}_6^{(k)},\quad k=1,\ldots,$$

where

$$\mathsf{E}_1^{(k)}=\left\{\Delta\le\hat{\mathcal{D}}_i^{(k)}(1/2)\le 2\Delta,\ i=0,1,2\right\}$$
$$\mathsf{E}_2^{(k)}=\left\{\hat{\mathcal{R}}_j^{(k)}(1/2)\le\Delta/2,\ j=1,2\right\}$$
$$\mathsf{E}_3^{(k)}=\left\{\sup_{1/2\le s\le 1}|\hat{D}_i^{(k)}(1/2,s)|\le\Delta/8,\ i=0,1,2\right\}$$

$$\mathsf{E}_4^{(k)} = \left\{ \sup_{1/2\le s\le 1} \hat{R}_j^{(k)}(1/2, s) \le \Delta/4,\ j = 1, 2 \right\}$$

$$\mathsf{E}_5^{(k)} = \left\{ \sup_{1/2\le s\le 1} \hat{\mathcal{D}}_i(s) \le 0,\ i = 3, \dots, m-1 \right\}$$

$$\mathsf{E}_6^{(k)} = \left\{ \inf_{1/2\le s\le 1} \hat{\mathcal{R}}_j(s) \ge 7\Delta,\ j = 3, \dots, n \right\}.$$

The definition implies that at any time $t \in [1/2, 1]$ and on any sample path in $\Omega^{(k)}$, for components $j = 1, 2$,

$$\begin{aligned}\tilde{Q}_j^{(k)}(t) &= [\hat{\mathcal{D}}_0^{(k)}(1/2) + \hat{D}_0^{(k)}(1/2, t)] + [\hat{\mathcal{D}}_j^{(k)}(1/2) \\ &\quad + \hat{D}_j^{(k)}(1/2, t)] - [\hat{\mathcal{R}}_j^{(k)}(1/2) + \hat{R}_j^{(k)}(1/2, t)] \\ &\ge \Delta\ (> 0),\end{aligned} \tag{32}$$

and for components $j = 3, \dots, n$ (recall that all entries in $\tilde{A}$ are binary),

$$\begin{aligned}\tilde{Q}_j^{(k)}(t) &= \sum_{i=0}^{2} a_{ji}[\hat{\mathcal{D}}_i^{(k)}(1/2) + \hat{D}_i^{(k)}(1/2, t)] \\ &\quad + \sum_{i=3}^{m-1} a_{ji}\hat{\mathcal{D}}_i^{(k)}(t) - \hat{\mathcal{R}}_j^{(k)}(t) \le 0.\end{aligned} \tag{33}$$

Given $\tilde{\mathbf{Q}}^{(\mathbf{k})}(t)$ in the above, A in (29), and $c_0 > c_1 + c_2$, (6) yields

$\tilde{B}_1^{(k)*}(t) = \tilde{Q}_1^{(k)+}(t)$, $\quad \tilde{B}_2^{(k)*}(t) = \tilde{Q}_2^{(k)+}(t)$, $\quad$ and
$\tilde{B}_i^{(k)*}(t) = 0 \quad (i = 0, 3, \dots, m-1)$.

Hence at any time $t \in [1/2, 1]$ and on any sample path in $\Omega^{(k)}$,

$$\begin{aligned}\tilde{B}_i^{(k)}(t) - \tilde{B}_i^{(k)*}(t) &= \tilde{B}_i^{(k)}(t) - \tilde{Q}_i^{(k)+}(t) \\ &= \tilde{B}_i^{(k)}(t) - [\tilde{B}_0^{(k)}(t) + \tilde{B}_i^{(k)}(t) - \tilde{I}_i^{(k)}(t)]^+ \\ &\ge -\tilde{B}_0^{(k)}(t)\end{aligned}$$

for $i = 1, 2$, and

$$\tilde{B}_i^{(k)}(t) - \tilde{B}_i^{(k)*}(t) = \tilde{B}_i^{(k)}(t)\ (\ge 0)$$

for $i = 0$ and $i = 3, \dots, m-1$. Since $\sum_{i=0}^{m-1} c_i[\tilde{B}_i^{(k)}(t) - \tilde{B}_i^{(k)*}(t)] \ge 0$ for each $t \ge 0$ and on every sample path, the above indicates

$$\begin{aligned}&\frac{\mathcal{C}^{(k)} - \underline{\mathcal{C}}^{(k)}}{\sqrt{k}} \\ &\ge E\left[\mathbf{1}(\Omega^{(k)}) \int_{1/2}^{1} e^{-\delta t} \sum_{i=0}^{m-1} c_i E\left[\tilde{B}_i^{(k)}(t) - \tilde{B}_i^{(k)*}(t)\right] dt\right] \\ &\ge E\left[\mathbf{1}(\Omega^{(k)}) \int_{1/2}^{1} e^{-\delta t}(c_0 - c_1 - c_2)\tilde{B}_0^{(k)}(t) dt\right],\end{aligned} \tag{34}$$

where $\mathbf{1}(\Omega^{(k)})$ is the indicator function that a sample path is in $\Omega^{(k)}$. Hence we can prove Theorem 2 by showing that

$$\lim_{k\to\infty} \mathbb{P}[\Omega^{(k)}] > 0, \tag{35}$$

and when k is sufficiently large,

$$\int_{1/2}^{1} e^{-\delta t}\tilde{B}_0^{(k)}(t) dt \ge e^{-\delta}\frac{\Delta}{4}, \quad \text{a.s.,} \tag{36}$$

in $\Omega^{(k)}$. These two conditions imply that

$$\lim_{k\to\infty} E\left[\mathbf{1}(\Omega^{(k)}) \int_{1/2}^{1} e^{-\delta t}\tilde{B}_0^{(k)}(t) dt\right] \ge e^{-\delta}\frac{\Delta}{4} \lim_{k\to\infty} \mathbb{P}[\Omega^{(k)}] > 0.$$

Since $c_0 > c_1 + c_2$, (31) follows immediately from the above inequality and (34).

To prove (35), since $\mathcal{D}^{(k)}(t)$ and $\mathcal{R}^{(k)}(t)$ $(t \ge 0)$ are independent Poisson Processes, $\mathsf{E}_q^{(k)}$ $(q = 1, 2, 3, 4, 5, 6)$ are independent of each other, so if

$$\lim_{k\to\infty} \mathbb{P}[\mathsf{E}_q^{(k)}] > 0, \quad q = 1, 2, 3, 4, 5, 6,$$

then (35) holds. Following (12),

$$\lim_{k\to\infty} \mathbb{P}[\mathsf{E}_1^{(k)}] = \prod_{i=0}^{2} \mathbb{P}[\Delta/\sigma_i \le \mathcal{B}(1/2) \le 2\Delta/\sigma_i]$$

$$\lim_{k\to\infty} \mathbb{P}[\mathsf{E}_2^{(k)}] = \prod_{j=1}^{2} \mathbb{P}[\mathcal{B}(1/2) \le \Delta/(2\gamma_j)].$$

Applying FCLT and Continuous Mapping Theorem,

$$\lim_{k\to\infty} \mathbb{P}[\mathsf{E}_3^{(k)}] = \prod_{i=0}^{2} \mathbb{P}[\sup_{0\le s\le 1/2} |\mathcal{B}(s)| \le \Delta/(8\sigma_i) | \mathcal{B}(0) = 0]$$

$$\lim_{k\to\infty} \mathbb{P}[\mathsf{E}_4^{(k)}] \ge \prod_{j=1}^{2} \mathbb{P}[\sup_{0\le s\le 1/2} |\mathcal{B}(s)| \le \Delta/(4\gamma_j) | \mathcal{B}(0) = 0]$$

$$\begin{aligned}\lim_{k\to\infty} \mathbb{P}[\mathsf{E}_5^{(k)}] &\ge \lim_{k\to\infty} \prod_{i=3}^{m-1} \mathbb{P}[\hat{\mathcal{D}}_i^{(k)}(1/2) \le -\Delta] \\ &\quad \times \mathbb{P}[\sup_{1/2\le s\le 1} |\hat{D}_i^{(k)}(1/2, s)| \le \Delta] \\ &= \prod_{i=3}^{m-1} \mathbb{P}[\mathcal{B}(1/2) \le -\Delta/\sigma_i] \\ &\quad \times \mathbb{P}[\sup_{0\le s\le 1/2} |\mathcal{B}(s)| \le \Delta/\sigma_i | \mathcal{B}(0) = 0]\end{aligned}$$

$$\begin{aligned}\lim_{k\to\infty} \mathbb{P}[\mathsf{E}_6^{(k)}] &\ge \lim_{k\to\infty} \prod_{j=3}^{n} \mathbb{P}[\hat{\mathcal{R}}_j^{(k)}(1/2) \ge 8\Delta] \\ &\quad \times \mathbb{P}[\sup_{1/2\le s\le 1} |\hat{R}_j^{(k)}(1/2, s)| \le \Delta] \\ &= \prod_{j=3}^{n} \mathbb{P}[\mathcal{B}(1/2) \ge 8\Delta/\gamma_j] \\ &\quad \times \mathbb{P}[\sup_{0\le s\le 1/2} |\mathcal{B}(s)| \le \Delta/\gamma_j | \mathcal{B}(0) = 0].\end{aligned}$$

The above implies that (35) holds because for any given constant $\xi > 0$,

$\mathbb{P}[\mathcal{B}(1/2) \ge \xi] > 0$, $\quad \mathbb{P}[\mathcal{B}(1/2) \le -\xi] > 0$,
$\mathbb{P}[\xi \le \mathcal{B}(1/2) \le 2\xi] > 0$, and $\mathbb{P}[\sup_{0\le s\le 1/2} |\mathcal{B}(s)| \le \xi | \mathcal{B}(0) = 0] > 0$.

To prove (36), define

$$t_0^{(k)} = \inf_{t\ge 1/2}\{t : \tilde{B}_1^{(k)}(t) \times \tilde{B}_2^{(k)}(t) = 0\}.$$

If $t_0^{(k)} > 3/4$, then for $t \in [1/2, 3/4]$, $\tilde{B}_1^{(k)}(t) > 0$, $\tilde{B}_2^{(k)}(t) > 0$, and thus by (30),

$$\tilde{I}_1^{(k)}(t) = \tilde{I}_2^{(k)}(t) = 0.$$

The above implies that during $[1/2, 3/4]$, demand 0 can be served only when components 1 and 2 arrive at the same time, which does not happen (a.s.) because $\mathcal{R}_1^{(k)}(t)$ and $\mathcal{R}_2^{(k)}(t)$ are independent Poisson processes. Therefore,

$$\tilde{B}_0^{(k)}(t) \geq \frac{D_0^{(k)}(1/2,t)}{\sqrt{k}} = \sqrt{k}\lambda_0(t-1/2) + \hat{D}_0^{(k)}(1/2,t)$$
$$\geq \sqrt{k}\lambda_0(t-1/2) - \frac{\Delta}{8}, \quad \text{a.s.}$$

for all $t \in [1/2, 3/4]$ (the last inequality holds because the sample path is in $\mathsf{E}_3^{(k)}$). When k is sufficiently large, (36) follows from

$$\int_{1/2}^{1} e^{-\delta t}\tilde{B}_0^{(k)}(t)dt \geq \int_{1/2}^{3/4} e^{-\delta t}\left[\sqrt{k}\lambda_0(t-1/2) - \frac{\Delta}{8}\right]dt$$
$$> e^{-\delta}\frac{\Delta}{4}.$$

For cases where $t_0^{(k)} \leq 3/4$, since $\tilde{B}_1^{(k)}(t_0^{(k)}) \wedge \tilde{B}_2^{(k)}(t_0^{(k)}) = 0$ and (32) applies,

$$\tilde{B}_0^{(k)}(t_0^{(k)}) \geq \tilde{Q}_1^{(k)}(t_0^{(k)}) \wedge \tilde{Q}_2^{(k)}(t_0^{(k)}) \geq \Delta. \quad (37)$$

Under (30), demand 0 is served only when it has a new arrival or one of components 1 or 2 is received. Let $\{\upsilon_1, \upsilon_2, \ldots\}$ be the set of these times during $[1/2, 1]$. Demand arrival and component production follow independent Poisson processes, so demand 1 or 2 does not arrive and components 1 and 2 are not simultaneously received at υ_l $(l = 1, \ldots,)$ (a.s.). Thus for demand 0 to be served,

$$\tilde{I}_1^{(k)}(\upsilon_l^-) \vee \tilde{I}_2^{(k)}(\upsilon_l^-) > 0, \quad \text{a.s.}, \ l = 1, 2, \ldots.$$

To satisfy (30), the above and (33) imply that

$$\tilde{B}_1^{(k)}(\upsilon_l^-) \wedge \tilde{B}_2^{(k)}(\upsilon_l^-) = 0, \quad \text{a.s.}, \ l = 1, 2, \ldots$$

which, with no arrival of demand 1 or 2 at υ_l (a.s.), implies that

$$\tilde{B}_1^{(k)}(\upsilon_l) \wedge \tilde{B}_2^{(k)}(\upsilon_l) = 0, \quad \text{a.s.}, \ l = 1, 2, \ldots.$$

Since $\upsilon_l \in [1/2, 1]$, (32) holds, and thus at every υ_l $(l = 1, 2, \ldots,)$,

$$\tilde{B}_0^{(k)}(\upsilon_l) \geq \max_{i=1,2}\{\tilde{Q}_i^{(k)}(\upsilon_l) - \tilde{B}_i^{(k)}(\upsilon_l)\}$$
$$\geq \inf_{1/2 \leq s \leq 1}\left\{\tilde{Q}_1^{(k)}(s) \wedge \tilde{Q}_2^{(k)}(s)\right\} \geq \Delta. \quad (38)$$

Since demand 0 is not served at $t \in [1/2, 1]$ when $t \neq \upsilon_l$ $(l = 1, \ldots)$, $\tilde{B}_0^{(k)}(t)$ does not decrease at these times. Hence (37) and (38) imply that

$$\tilde{B}_0^{(k)}(t) \geq \Delta, 3/4 \leq t \leq 1, \quad \text{a.s.},$$

and (36) follows immediately. ■

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