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On the core of cooperative queueing games

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ABSTRACT

We consider a class of cooperative games for managing several canonical queueing systems. When cooperating parties invest optimally in common capacity or choose the optimal amount of demand to serve, cooperation leads to "single-attribute" games whose characteristic function is embedded in a one-dimensional function. We show that when and only when the latter function is elastic will all embedded games have a non-empty core, and the core contains a population monotonic allocation. We present sufficient conditions for this property to be satisfied. Our analysis reveals that in most Erlang B and Erlang C queueing systems, the games under our consideration have a non-empty core, but there are exceptions, which we illustrate through a counterexample.

1. Introduction

Many service systems exhibit economies of scale, which encourage operators to cooperate by sharing resources or pooling demands. The cooperation is sustainable only if participants derive a higher reward than the amount they would get by not cooperating or cooperating only with a subgroup of peers. Such reward is feasible if and only if the corresponding cooperative game has a non-empty core.

Queueing models are common representations of service systems, and past studies have identified various ways to test non-emptiness of the core in cooperative games built upon these models. Garcial-Sanz et al. [3] and Yu et al. [10] prove non-emptiness of the core by showing that some particular allocation scheme is an element. Gonzalez and Herrero [4] show that their game has a non-empty core because its characteristic function is the sum of characteristic functions of other games, the core of which is obviously non-empty. Anily and Haviv [1] make a clever use of a well-known result that convex games always have a non-empty core and prove that the core of their game is non-empty because it is a super-set of the core of a convex game. In this paper, we introduce another approach for a class of games in which operators not only cooperate but also optimize a part of their joint operations. As a consequence of this optimization, the characteristic function of cooperative games in many queueing systems can be derived from a particular one-dimensional function. We show that a property of this function determines the non-emptiness of the core, and apply this result to several canonical queueing models. In a recent paper, Karsten et al. [5] present a different approach to study cooperation in one of these queueing models.

Our discussion proceeds as follows: we introduce some definitions in Section 2, present two key theorems in Section 3 and apply them to analyse cooperative games in queueing systems in Section 4.

2. Definitions

Let $\Gamma(N, v)$ be a cooperative game of transferable utility, where $N = \{1, \ldots, n\}$ is the set of players and $v = \{v(S), S \subseteq N\}$, with $v(\emptyset) = 0$, is the characteristic function that specifies utilities achievable by subsets of players. Without loss of generality, we assume that higher value of v is preferred. Hence the core of the game is non-empty if there exists some $\{r_i, i \in N\}$ such that

$$\sum_{i\in N} r_i = v(N) \quad \text{and} \quad \sum_{i\in S} r_i \geq v(S) \quad \text{for all } S \subseteq N. \tag{1}$$

Bondareva [2] and Shapley [7] independently made a general characterization of cooperative games with a non-empty core by showing that a game has a non-empty core if and only if the game is balanced.

For a given game $\Gamma(N, v)$, $\{\omega_i(S), S \subseteq N, i \in S\}$ is a population monotonic allocation scheme if for all $S \subseteq N$,

$$\sum_{i\in S} \omega_i(S) = v(S),$$

and for all $i \in S \subset N$,

$$\omega_i(S) \leq \omega_i(S') \quad \text{if } S \subseteq S'.$$

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One may verify that $r_i = \omega_i(N)$ $(i \in N)$ satisfies (1), and thus is an element of the core (see e.g., [9]).

Let $\pi(a)$ be a non-negative function defined on $\mathbf{R}_+$ (or $\mathbf{Z}_+$) with $\pi(0) = 0$. We consider $\pi(a)$ to be elastic if

$$\frac{\pi(a_1)}{a_1} \leq \frac{\pi(a_2)}{a_2} \quad \text{for all } a_1, a_2 \text{ such that } a_1 \leq a_2. \tag{2}$$

Intuitively, if $\pi(a)$ is differentiable over $\mathbf{R}_+$, then

$$\varepsilon(a) = \frac{\mathrm{d}\pi(a)/\pi(a)}{\mathrm{d}a/a}$$

is referred to as elasticity of $\pi(a)$ in the economics literature, and the function $\pi(a)$ is considered to be elastic if $\varepsilon(a) \geq 1$ for all a, which is equivalent to (2).

We define a game $\Gamma(N, v)$ to be a single-attribute game if each player $i \in N$ is endowed with an amount $a_i \in \mathbf{R}_+$ (or $\mathbf{Z}_+$) of a single entity referred to as the attribute and the characteristic function $v(S)(S \subseteq N)$ is determined by

$$v(S) = \pi(\bar{a}_S) \quad \text{for all } S \subseteq N,$$

where

$$\bar{a}_S = \sum_{i \in S} a_i.$$

In this case we say that the single-attribute game $\Gamma(N, v)$ is embedded in the function $\pi(a)$.

Let $\mathbf{x}$ be a m dimensional vector (including the special case $m = 1$) and $f(\mathbf{x})$ be a function on $\mathcal{D}_{\mathbf{X}} \subseteq \mathbf{R}_+^m$. As is commonly defined, $f(\mathbf{x})$ is subhomogeneous (superhomogeneous) of degree l if $f(\phi\mathbf{x}) \leq \phi^l f(\mathbf{x})$ $(f(\phi\mathbf{x}) \geq \phi^l f(\mathbf{x}))$ for any $\phi \geq 1$ such that $\phi\mathbf{x} \in \mathcal{D}_{\mathbf{X}}$.

3. Key theorems

We consider a family of single-attribute games $\Gamma(N, v)$ embedded in the function $\pi(a)$. Our first theorem relates the elasticity of $\pi(a)$ with the core of $\Gamma(N, v)$.

Theorem 1. *If $\pi(a)$ is elastic, then every single-attribute game $\Gamma(N, v)$ embedded in $\pi(a)$ has a non-empty core that contains a population monotonic allocation. If $\pi(a)$ is not elastic, then there exists a single-attribute game $\Gamma(N, v)$ embedded in $\pi(a)$ that has an empty core.*

Proof. If $\pi(a)$ is elastic, then

$$\omega_i(S) = a_i \frac{v(S)}{\bar{a}_S} = a_i \frac{\pi(\bar{a}_S)}{\bar{a}_S} \quad (S \subseteq N, i \in S) \tag{3}$$

is a population monotonic allocation. So $r_i = \omega_i(N)$ $(i \in N)$ is an element of the core.

If $\pi(a)$ is not elastic, there exist a_1 and a_2 such that $a_1 < a_2$ and

$$\frac{\pi(a_1)}{a_1} > \frac{\pi(a_2)}{a_2}. \tag{4}$$

If both a_1 and a_2 are rational numbers, then for $i = 1, 2$,

$$a_i = \frac{b_i}{d_i}, \quad \text{where } b_i \text{ and } d_i \text{ are mutual primes.}$$

Let $\bar{d}$ be the minimum common multiple of d_1 and d_2, and n_1 and n_2 be integers such that

$$n_1 = a_1\bar{d} \quad \text{and} \quad n_2 = a_2\bar{d}.$$

(Note that $\bar{d} = 1$ if a_1 and a_2 are both integers.) Then from (4),

$$\frac{\pi(n_1/\bar{d})}{n_1} = \frac{\pi(a_1)}{a_1\bar{d}} > \frac{\pi(a_2)}{a_2\bar{d}} = \frac{\pi(n_2/\bar{d})}{n_2},$$

so there exists some $n \in \{n_1 + 1, \ldots, n_2\}$ such that

$$\frac{\pi((n-1)/\bar{d})}{n-1} > \frac{\pi(n/\bar{d})}{n}. \tag{5}$$

Consider a single-attribute game $\Gamma(N, v)$ embedded in $\pi(a)$, where

$$N = \{1, \ldots, n\}, \qquad a_1 = \cdots = a_n = \frac{1}{\bar{d}}, \quad \text{and}$$
$$v(S) = \pi(|S|/\bar{d}) \quad (S \subseteq N).$$

We can show the game has an empty core by contradiction: suppose $\{r_i, i \in N\}$ is an element of the core, then

$$v(N \setminus \{i\}) \leq v(N) - r_i, \quad i \in N,$$

which by the embedding in π yields

$$\pi\left(\frac{n-1}{\bar{d}}\right) \leq \pi\left(\frac{n}{\bar{d}}\right) - r_i.$$

Summing up the above over all i and using $v(N) = r_1 + \cdots + r_n$, yields

$$n\pi\left(\frac{n-1}{\bar{d}}\right) \leq n\pi\left(\frac{n}{\bar{d}}\right) - \pi\left(\frac{n}{\bar{d}}\right) = (n-1)\pi\left(\frac{n}{\bar{d}}\right),$$

which violates (5).

If either a_1 or a_2 is an irrational number, then by (4)

$$\eta = a_2\pi(a_1) - a_1\pi(a_2) > 0.$$

If $\pi(a)$ is monotonically increasing in a, $a_1 + \delta \leq a_2 - \delta$, where $\delta = \eta/(\pi(a_1) + \pi(a_2)) > 0$. Since the set of rational numbers is a dense subset of the set of real numbers, there exists rational numbers a_1' and a_2' such that

$$a_1 < a_1' < a_1 + \delta \leq a_2 - \delta < a_2' < a_2.$$

Moreover,

$$\frac{\pi(a_1')}{a_1'} - \frac{\pi(a_2')}{a_2'} > \frac{\pi(a_1)}{a_1 + \delta} - \frac{\pi(a_2)}{a_2 - \delta} = \frac{\eta - \delta(\pi(a_1) + \pi(a_2))}{(a_1 + \delta)(a_2 - \delta)} = 0.$$

Applying the above analysis for rational numbers to a_1' and a_2', there exists some $\Gamma(N, v)$ embedded in $\pi(a)$ that has an empty core.

If $\pi(a)$ is not monotonically increasing in a, then there exist $a' < a''$ such that $\pi(a') > \pi(a'')$. Let $a_1 = a'$ and $a_2 = a'' - a'$. It is easy to see that the game $\Gamma(N, v)$ where $N = \{1, 2\}$ and

$$v(\{1\}) = \pi(a_1), \qquad v(\{2\}) = \pi(a_2) \quad \text{and}$$
$$v(\{1, 2\}) = \pi(a_1 + a_2),$$

has an empty core. □

The theorem indicates that the condition $\pi(a)$ being elastic is not only sufficient for existence of a non-empty core but also for cooperation to be self-enforcing. All games embedded in an elastic function have a population monotonic allocation rule that allows the reward to grow with the number of participants. In fact, this population monotonic allocation is in very simple proportional form as described by (3).

In the context of this paper, $\pi(a)$ corresponds to either the profit or social welfare of operating a queueing system. To discuss situations in which the function is elastic, we define

$$\Pi(x, y) = U(x)[1 - \gamma_1 B(x, y)] - pxQ(x, y) - C(y)$$

where

$$x \in \mathbf{R}_+, \qquad y \in \mathcal{D}_Y \subseteq \mathbf{R}_+, \qquad p \geq 0, \qquad \gamma_1, \gamma_2 \in \{0, 1\},$$
$$\text{and} \quad \gamma_2 Q(x, y) \leq T.$$

To attach proper meaning to the above terms, consider a queueing system with Poisson arrivals. Then x may represent demand for service (arrival rate); y may represent system capacity, either as the service rate ($\mathcal{D}_Y = \mathbf{R}_+$) or as the number of servers ($\mathcal{D}_Y = \mathbf{Z}_+$). The value of service is $U(x)$ per unit of time, which is counted if the arrival is not blocked. The cost of operating system capacity is $C(y)$ per unit of time. The blocking probability and the relevant performance measure are $B(x, y)$ and $Q(x, y)$ respectively ($B(x, y) = Q(x, y)$ if the blocking probability is the performance measure of interest). Blocking systems are represented by indicator value $\gamma_1 = 1$ and systems with guaranteed performance target T are represented by indicator value $\gamma_2 = 1$. Poor performance is penalized by a cost p per customer.

We assume that the following conditions apply to $\Pi(x, y)$:

1. $U(x) \geq 0$; $U(x)$ is (weakly) superhomogeneous of degree 1.
2. $0 \leq B(x, y) \leq 1$; $B(x, y)$ is subhomogeneous of degree 0.
3. $Q(x, y) \geq 0$; $Q(0, y) = 0$ for all y and $Q(x, +\infty) = 0$ for all x; $Q(x, y)$ is subhomogeneous of degree 0.
4. $C(y) > 0$; $C(y)$ is (weakly) subhomogeneous of degree 1.

As we show below, conditions on $B(x, y)$ and $Q(x, y)$ apply to many commonly considered queueing systems and performance measures. It is also quite reasonable to expect $U(x)$ to be weakly superhomogeneous and $C(y)$ to be weakly subhomogeneous of degree 1. For instance, if a profit-driven operator collects a fixed price per customer and pays a fixed cost for each unit of capacity, then $U(x)$ and $C(y)$ are linear functions that satisfy these properties.

Theorem 2. *Suppose $\Pi(x, y)$ satisfies conditions* 1–4. *If for any $y \in \mathbf{R}_+$ (or $\mathbf{Z}_+$),*

$$\pi(y) = \sup_{x\in\mathbf{R}_+} \{\Pi(x, y) | \gamma_2 Q(x, y) \leq T\} \tag{6}$$

has a finite optimizer, then $\pi(y)$ is elastic. Similarly, if for any $x \in \mathbf{R}_+$,

$$\bar{\pi}(x) = \sup_{y\in\mathcal{D}_Y} \{\Pi(x, y) | \gamma_2 Q(x, y) \leq T\} \tag{7}$$

has a finite optimizer, then $\bar{\pi}(x)$ is also elastic if $\mathcal{D}_Y = \mathbf{R}_+$.

Proof. Under the assumption of the theorem, $\pi(y)$ is well defined over $\mathbf{R}_+$ (or $\mathbf{Z}_+$). Suppose x_1^* maximizes $\Pi(x, y_1)$. For any $y_2 > y_1$, let

$$\phi = \frac{y_2}{y_1} > 1,$$

then $\phi x_1^* \in \mathbf{R}_+$, and because $Q(x, y)$ is subhomogeneous of degree 0,

$$Q(\phi x_1^*, y_2) = Q(\phi x_1^*, \phi y_1) \leq Q(x_1^*, y_1) \leq T.$$

Hence ϕx_1^* is a feasible solution for optimizing $\Pi(x, y_2)$, so $\pi(y_2) \geq \Pi(\phi x_1^*, y_2)$. Under the assumed above superhomogeneous and subhomogeneous conditions,

$$\begin{aligned}\frac{\pi(y_2)}{y_2} &\geq \frac{\Pi(\phi x_1^*, y_2)}{y_2} \\ &= \frac{U(\phi x_1^*)}{\phi y_1}[1 - \gamma_1 B(\phi x_1^*, \phi y_1)] \\ &\quad - p x_1^* \frac{Q(\phi x_1^*, \phi y_1)}{y_1} - \frac{C(\phi y_1)}{\phi y_1} \\ &\geq \frac{U(x_1^*)}{y_1}[1 - \gamma_1 B(x_1^*, y_1)] - p x_1^* \frac{Q(x_1^*, y_1)}{y_1} - \frac{C(y_1)}{y_1} \\ &= \frac{\pi(y_1)}{y_1}.\end{aligned}$$

A similar proof applies to the result that $\bar{\pi}(x)$ is elastic when $\mathcal{D}_Y = \mathbf{R}_+$. □

4. Application to games of managing queueing systems

We apply the above theorems to cooperative games arising in classical Erlang B and Erlang C queueing systems. We divide these systems into two types. A type-μ system has a fixed number of identical servers and its capacity can be varied by changing the service rate. A type-K system has a fixed service rate (which we normalize to 1) and its capacity is controlled by changing the number of servers. For each type, we consider two modes of cooperation. Under resource sharing mode, operators combine their service capacities and optimize the amount of demand to be served jointly. Under demand pooling mode, operators combine their customer bases and optimize the system capacity that they jointly invest in. We use the arrival rate λ to represent the previously defined demand variable x (e.g., x in $\Pi(x, y)$) and the service rate μ in type-μ cases and the number of servers K in type-K cases to represent the previously defined capacity variable y.

4.1. Type-μ systems

Under this category, we consider Erlang B models with a general service time and Erlang C models with an exponentially distributed service time. Let K denote the number of servers. In the Erlang B case, a natural performance measure is the blocking probability, which is determined as follows:

$$B(\lambda, \mu) = \sigma^{-1} \frac{\rho^K}{K!}, \quad \text{where } \rho = \lambda/\mu \quad \text{and} \quad \sigma = \sum_{k=0}^{K} \frac{\rho^k}{k!}.$$

Clearly $B(\lambda, \mu)$ is subhomogeneous of degree 0.

In Erlang C models, there is no blocking, and the performance is commonly measured by the probability of waiting, in which case

$$Q(\lambda, \mu) = \frac{\rho^K/K!}{(1 - \rho/K)\sigma + \rho/K(\rho^K/K!)} \tag{8}$$

or the expected waiting time, in which case

$$Q(\lambda, \mu) = \frac{\rho^K/K!}{[(1 - \rho/K)\sigma + \rho/K(\rho^K/K!)](K\mu - \lambda)} \tag{9}$$

or the probability that the waiting time exceeds some threshold τ, in which case

$$Q(\lambda, \mu) = \frac{\rho^K/K!}{(1 - \rho/K)\sigma + \rho/K(\rho^K/K!)} e^{-(K\mu-\lambda)\tau}. \tag{10}$$

Since ρ and σ are both homogeneous of degree 0, in all three cases it can be shown that $Q(\lambda, \mu)$ is subhomogeneous of degree 0. Summarizing the above leads to the following corollary to Theorems 1 and 2.

Corollary 3. *Among type-μ systems, both $\pi(\mu)$ in (6) and $\bar{\pi}(\lambda)$ in (7) obtained for the Erlang B model are elastic if the blocking probability is used as the performance measure (i.e., $Q(\lambda, \mu) = B(\lambda, \mu)$). The same is true for the Erlang C model if the performance measure is defined by (8), (9), or (10). Under these circumstances, any resource sharing game embedded in $\pi(\mu)$ and any demand pooling game embedded in $\bar{\pi}(\lambda)$ has a non-empty core.*

4.2. Type-K systems

Similar to the discussion of type-μ systems, with the service rate normalized to 1, the blocking probability for the Erlang B model is

$$B(\lambda, K) = \tilde{\sigma}^{-1} \frac{\lambda^K}{K!}, \quad \text{where } \tilde{\sigma} = \sum_{k=0}^{K} \frac{\lambda^k}{k!}. \tag{11}$$

Applying (11) to three performance measures defined in (8)–(10) and rearranging terms (note that $\rho = \lambda$ when $\mu = 1$), the probability of waiting as a function of λ and K is

$$Q(\lambda, K) = \frac{1}{(1-\lambda/K)/B(\lambda, K) + \lambda/K}, \tag{12}$$

the expected waiting time is

$$Q(\lambda, K) = \frac{1}{[(1-\lambda/K)/B(\lambda, K) + \lambda/K](K-\lambda)} \tag{13}$$

and the probability that the waiting time exceeds some threshold τ is

$$Q(\lambda, K) = \frac{e^{-(K-\lambda)\tau}}{(1-\lambda/K)/B(\lambda, K) + \lambda/K}. \tag{14}$$

It was shown by Smith and Whitt [8] that $B(\lambda, K)$ in (11) is subhomogeneous of degree 0. Applying this result to the second equality in (12)–(14), in all three cases, $Q(\lambda, K)$ is subhomogeneous of degree 0. Therefore,

Corollary 4. *Among type-K systems, $\pi(K)$ in (6) for the Erlang B model is elastic if the blocking probability is used as the performance measure ($Q(\lambda, K) = B(\lambda, K)$). The function is also elastic for the Erlang C model if the performance measure $Q(\lambda, K)$ is given by (12), (13), or (14). Under these circumstances, any resource sharing game embedded in $\pi(K)$ has a non-empty core.*

Unlike the case with type-μ systems, here we cannot make a categorical claim that demand pooling games embedded in $\bar{\pi}(\lambda)$ always have a non-empty core, for either the Erlang B or Erlang C model. Since the number of servers (K) takes integer values, the sufficient condition $\mathcal{D}_Y = \mathbf{R}_+$ in Theorem 2 is absent. Consequently, depending on the system details, $\bar{\pi}(\lambda)$ may or may not be elastic, which we illustrate by the following two examples. In both cases, we consider the Erlang B model and use the blocking probability as the performance measure.

Example 1. The following theorem illustrate a situation in which demand pooling games embedded in $\bar{\pi}(\lambda)$ always have a non-empty core. A similar result to this theorem is shown by Karsten et al. [5] using a different method, namely an extension of the Erlang loss function.

Theorem 5. *Suppose $B(\lambda, K)$ is the blocking probability in an Erlang B model and used as the performance measure ($Q(\lambda, K) = B(\lambda, K)$). Suppose there is no guaranteed performance target (i.e., $\gamma_2 = 0$) and both utility and cost are linear functions (i.e, $U(\lambda) = \upsilon\lambda$ and $C(K) = cK$). Then*

$$\bar{\pi}(\lambda) = \sup_{K\in\mathbf{Z}_+} \{\Pi(\lambda, K) = \upsilon\lambda[1 - B(\lambda, K)] - p\lambda B(\lambda, K) - cK\} \tag{15}$$

is elastic. Thus any demand pooling game embedded in $\bar{\pi}(\lambda)$ has a non-empty core.

Proof. Fix λ and let $K \to \infty$. Then $B(\lambda, K) \to 0$, so that $\Pi(\lambda, K) \to -\infty$. Thus the right-hand side of (15) has a finite maximizer. Let $\mathcal{K}^*(\lambda)$ denote the (finite) set of maximizers in (15), and let $|\mathcal{K}^*(\lambda)|$ denote the cardinality of $\mathcal{K}^*(\lambda)$. If $|\mathcal{K}^*(\lambda)| \ge 2$ then we can denote two of the maximizers as K and $K + j$ for appropriate $K \ge 0, j > 0$ and we have $\Pi(\lambda, K) = \Pi(\lambda, K + j)$. Let

$$\Delta(\lambda, K) = \Pi(\lambda, K) - \Pi(\lambda, K + 1), \quad K \ge 0,$$

so that

$$\Delta(\lambda, K) = c - (\upsilon + p)\lambda[B(\lambda, K) - B(\lambda, K + 1)].$$

We can write

$$\Pi(\lambda, K) - \Pi(\lambda, K + j) = \sum_{l=0}^{j-1} \Delta(\lambda, K + l).$$

It is well known that $B(\lambda, K) > B(\lambda, K + 1)$ for any $K \ge 0$. Further, by Theorem 1 of Messerli [6], $B(\lambda, K) - B(\lambda, K + 1)$ is strictly decreasing in K for $K \ge 0$. Thus, if $j > 1$, $\Delta(\lambda, K) < \Delta(\lambda, K + j - 1)$ so that $\Pi(\lambda, K) = \Pi(\lambda, K + j)$ implies that, if $j > 1$ then $\Delta(\lambda, K) < 0$ and $\Delta(\lambda, K + j - 1) > 0$. In this case $\Pi(\lambda, K + j - 1) > \Pi(\lambda, K + j)$, which contradicts the assumption that $K + j \in \mathcal{K}^*(\lambda)$. Thus $j > 1$ is impossible: $|\mathcal{K}^*(\lambda)| \le 2$, and when $|\mathcal{K}^*(\lambda)| = 2$, we have $\mathcal{K}^*(\lambda) = \{K, K + 1\}$ for some $K \ge 0$.

Note that, for each fixed K, $\lambda^{-1}\Pi(\lambda, K)$ is a continuously differentiable function of λ for $0 < \lambda < \infty$. We next show that, if $K^* \in \mathcal{K}^*(\lambda)$ then

$$\frac{\mathrm{d}(\Pi(\lambda, K^*)/\lambda)}{\mathrm{d}\lambda} \ge 0. \tag{16}$$

Consider $\lambda > 0$ and let K^* be a finite optimizer of $\Pi(\lambda, K)$ in (15), so

$$\begin{aligned} 0 &\le [\Pi(\lambda, K^*) - \Pi(\lambda, K^* + 1)]/\lambda \\ &= \frac{c}{\lambda} - (\upsilon + p)[B(\lambda, K^*) - B(\lambda, K^* + 1)]. \end{aligned} \tag{17}$$

We show by contradiction that for (17) to hold, (16) must also hold.

Suppose (16) does not hold, then

$$\frac{\mathrm{d}(\Pi(\lambda, K^*)/\lambda)}{\mathrm{d}\lambda} = -(\upsilon + p)\frac{\mathrm{d}B(\lambda, K^*)}{\mathrm{d}\lambda} + c\frac{K^*}{\lambda^2} < 0. \tag{18}$$

As a property of the Erlang B formula,

$$\frac{\mathrm{d}B(\lambda, K^*)}{\mathrm{d}\lambda} = B(\lambda, K^*)\left[\frac{K^*}{\lambda} - 1 + B(\lambda, K^*)\right],$$

which when applied to (18), leads to

$$c\frac{K^*}{\lambda^2} - (\upsilon + p)B(\lambda, K^*)\left[\frac{K^*}{\lambda} - 1 + B(\lambda, K^*)\right] < 0.$$

Using the above to substitute c/λ in (17), replacing $B(\lambda, K^* + 1)$ with

$$B(\lambda, K^* + 1) = \frac{\lambda B(\lambda, K^*)}{K^* + 1 + \lambda B(\lambda, K^*)},$$

which is also a property of Erlang B formula, and cancelling out the term $(\upsilon + p)B(\lambda, K^*)$,

$$\frac{\lambda}{K^*}\left(\frac{K^*}{\lambda} - 1 + B(\lambda, K^*)\right) > 1 - \frac{\lambda}{K^* + 1 + \lambda B(\lambda, K^*)},$$

which can be further simplified to

$$(K^* + 1)B(\lambda, K^*) + \lambda B^2(\lambda, K^*) - \lambda B(\lambda, K^*) > 1. \tag{19}$$

We show the contradiction by proving (19) cannot hold: following Erlang B formula, given K, λ

$$B(\lambda, K) = \lambda^K/K!\left(1 + \cdots + \frac{\lambda^K}{K!}\right)^{-1} = H^{-1}(\lambda, K)$$

where

$$H(\lambda, K) = \sum_{k=0}^{K} \frac{K!}{(K-i)!}\lambda^{-k} = 1 + \frac{K}{\lambda}H(K - 1, \lambda). \tag{20}$$

So (19) cannot hold if

$$H^2(\lambda, K) \ge (K + 1)H(\lambda, K) + \lambda - \lambda H(\lambda, K),$$

which, by using the second equality of (20), is the same as

$$\frac{H^2(\lambda, K)}{\lambda} \ge H(\lambda, K + 1) - H(\lambda, K). \tag{21}$$

Applying (20) to the right-hand side of (21),

$$H(\lambda, K+1) - H(\lambda, K) = \sum_{k=0}^{K}\left[\frac{(K+1)!}{(K+1-k)!} - \frac{K!}{(K-k)!}\right]\lambda^{-k} + (K+1)!\lambda^{-(K+1)} = K!\sum_{k=1}^{K+1}\frac{k}{(K+1-k)!}\lambda^{-k}. \quad (22)$$

Expanding the left-hand side of (21)

$$\frac{H^2(\lambda, K)}{\lambda} = \left(\sum_{k=0}^{K}\frac{K!}{(K-k)!}\lambda^{-k-1}\right)\left(\sum_{k=0}^{K}\frac{K!}{(K-k)!}\lambda^{-k}\right) = \sum_{k=1}^{2K+1}\lambda^{-k}\sum_{i=1}^{k}\frac{K!}{(K+i-k)!}\frac{K!}{(K+1-i)!} > K!\sum_{k=1}^{K+1}\lambda^{-k}\sum_{i=1}^{k}\frac{K!}{(K+i-k)!(K+1-i)!}. \quad (23)$$

For all $i = 1, \ldots, k$, $K - (K + i - k) = (K + 1 - i) - (K + 1 - k)$ and $K \geq K + 1 - i$, so

$$\frac{K!}{(K+i-k)!(K+1-i)!} \geq \frac{1}{(K+1-k)!},$$

which when applied to (22) and (23), leads to

$$\frac{H^2(\lambda, K)}{\lambda} \geq K!\sum_{k=1}^{K+1}\lambda^{-k}\frac{k}{(K+1-k)!} = H(\lambda, K+1) - H(\lambda, K),$$

which is the same as (21) and thus invalidates (19).

If $|\mathcal{K}^*(\lambda)| = 1$ then $\bar{\pi}$ is differentiable at λ, and by the above

$$\frac{\mathrm{d}(\bar{\pi}(\lambda)/\lambda)}{\mathrm{d}\lambda} \geq 0.$$

If $|\mathcal{K}^*(\lambda)| = 2$, look at $\mathrm{d}(\Pi(\lambda, K)/\lambda)/\mathrm{d}\lambda$ for $K \in \mathcal{K}^*(\lambda)$. Label the optimizers K' and K'' such that

$$\mathrm{d}(\Pi(\lambda, K')/\lambda)/\mathrm{d}\lambda \geq \mathrm{d}(\Pi(\lambda, K'')/\lambda)/\mathrm{d}\lambda.$$

If $\mathrm{d}(\Pi(\lambda, K')/\lambda)/\mathrm{d}\lambda = \mathrm{d}(\Pi(\lambda, K'')/\lambda)/\mathrm{d}\lambda$, then

$$\mathrm{d}(\bar{\pi}(\lambda)/\lambda)/\mathrm{d}\lambda = \mathrm{d}(\Pi(\lambda, K')/\lambda)/\mathrm{d}\lambda = \mathrm{d}(\Pi(\lambda, K'')/\lambda)/\mathrm{d}\lambda \geq 0.$$

If $\mathrm{d}(\Pi(\lambda, K')/\lambda)/\mathrm{d}\lambda > \mathrm{d}(\Pi(\lambda, K'')/\lambda)/\mathrm{d}\lambda$, by differentiability of $\Pi(\lambda, K)/\lambda$, for δ small enough, $|\mathcal{K}^*(\lambda + \epsilon)| = |\mathcal{K}^*(\lambda - \epsilon)| = 1$ for $0 < \epsilon < \delta$ and

$$\mathcal{K}^*(\lambda + \epsilon) = K', \quad 0 < \epsilon < \delta,$$
$$\mathcal{K}^*(\lambda - \epsilon) = K'', \quad 0 < \epsilon < \delta.$$

Thus the derivative of $\bar{\pi}(\lambda)/\lambda$ from the right is larger than the derivative from the left, so $\bar{\pi}$ is elastic. □

Example 2. Consider also an example of the Erlang B model with linear utility and cost functions but a slight difference from the above: instead of incurring a penalty cost p per blocked customer, we impose an upper limit T. Then

$$\bar{\pi}(\lambda) = \max_{K \in \mathbf{Z}_+}\{\Pi(\lambda, K) = \upsilon\lambda[1 - B(\lambda, K)] - cK | B(\lambda, K) \leq T\}.$$

Let

$$T = 0.1, \qquad \upsilon = 25, \quad \text{and} \quad c = 1.$$

Then $\bar{\pi}(\lambda)$ is not elastic as one may verify that

$$\bar{\pi}(0.05) = 25 * 0.05 * [1 - B(0.05, 1)] - 1 = 0.19,$$
$$\bar{\pi}(0.1) = 25 * 0.1 * [1 - B(0.1, 1)] - 1 = 1.27,$$
$$\bar{\pi}(0.15) = 25 * 0.15 * [1 - B(0.15, 2)] - 2 = 1.71.$$

So

$$\bar{\pi}(0.15)/0.15 \leq \bar{\pi}(0.1)/0.1.$$

By Theorem 1, there exists a single-attribute game embedded in $\bar{\pi}(K)$ that has an empty core. Such game is easy to construct: let $N = \{1, 2, 3\}$, and suppose each operator owns a demand of arrival rate 0.05. Hence in case of demand pooling,

$$\upsilon(\{1\}) = \upsilon(\{2\}) = \upsilon(\{3\}) = \bar{\pi}(0.05) = 0.19,$$
$$\upsilon(\{1, 2\}) = \upsilon(\{2, 3\}) = \upsilon(\{1, 3\}) = \bar{\pi}(0.1) = 1.27,$$
$$\upsilon(\{1, 2, 3\}) = \bar{\pi}(0.15) = 1.71.$$

It is easy to verify that this game has an empty core.

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