Electronic Companion

This e-companion is composed of three parts. Part I proves statements, equations, lemmas, and theorems in the paper. Part II presents the periodic-review (discrete time) formulation of our model as a supplement to the continuous-review formulation in the paper. Part III discusses a simple algorithm for solving our SP as a continuous optimization problem. Equations in the companion are referred by “EC-xx” and equations in the paper are referred by their numbers without the prefix “EC”.

I. Proofs of Theorems

Proof of Theorem 2.1

We prove that, for any $t \geq L$,

$$C^* s \preceq \mathbb{E} \left[ \sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \right]. \quad \text{(EC-1)}$$

By the definition of $C^{\gamma,p}$ in (8), this yields the claimed result. By (1) and (2),

$$\begin{align*}
\sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \\
= \sum_{i=1}^{m} b_i (B_i(t-L) + D_i(t) - Z_i(t)) + \sum_{j=1}^{n} h_j (I_j(t-L) + R_j(t-L)) - \sum_{i=1}^{m} a_{ij} Z_i(t) \\
= \sum_{i=1}^{m} b_i (D_i(t) + B_i(t-L)) + \sum_{j=1}^{n} h_j (I_j(t-L) + R_j(t-L)) - \sum_{i=1}^{m} c_i Z_i(t).
\end{align*} \quad \text{(EC-2)}$$

We need to introduce additional notation for the proof. In particular, we need to introduce the $\sigma$-algebra

$$\mathcal{F}_t = \sigma\{I(0^-), B(0^-); R(s), -L \leq s \leq t; D(s), Z(s), 0 \leq s \leq t\},$$

which represents the information available after any decisions at time $t$ have been made. It follows that

$$\mathbb{E} [B_i(t-L)|\mathcal{F}_{t-L}] = B_i(t-L), \quad 1 \leq i \leq m. \quad \text{(EC-3)}$$

Similarly,

$$\mathbb{E} [I_j(t-L)|\mathcal{F}_{t-L}] = I_j(t-L) \quad \text{and} \quad \mathbb{E} [R_j(t-L)|\mathcal{F}_{t-L}] = R_j(t-L), \quad 1 \leq j \leq n.$$ 

In addition, since $\{D(t), t \geq 0\}$ is a compound Poisson process, $D(t)$ is independent of $\mathcal{F}_{t-L}$ so that, conditioned on $\mathcal{F}_{t-L}$, $D(t)$ is equal in distribution to $D$, where $D$ is defined as a random variable that has the same distribution as $D(t)$ and is independent of $\{D(t), t \geq 0\}$. Taking the conditional expectation of (EC-2) with respect to $\mathcal{F}_{t-L}$ and imposing the feasibility conditions (9)
and (10) on \( Z(t) \), we thus obtain

\[
E \left[ \sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \bigg| \mathcal{F}_{t-L} \right]
\]

\[
= E \left[ \sum_{i=1}^{m} b_i B_i(t-L) + \sum_{i=1}^{m} b_i D_i(t) + \sum_{j=1}^{n} h_j (I_j(t-L) + R_j(t-L)) - \sum_{i=1}^{m} c_i Z_i(t) \bigg| \mathcal{F}_{t-L} \right]
\]

\[
= \sum_{i=1}^{m} b_i B_i(t-L) + \sum_{j=1}^{n} h_j (I_j(t-L) + R_j(t-L)) + E \left[ \sum_{i=1}^{m} b_i D_i(t) \right] - E \left[ \sum_{i=1}^{m} c_i Z_i(t) \bigg| \mathcal{F}_{t-L} \right]
\]

\[
\geq C_s (I(t-L) + R(t-L), B(t-L))
\]

\[
\geq C_s^*,
\]

where the first inequality comes from substituting \( I_j(t-L) + R_j(t-L) \) for \( y_j, 1 \leq j \leq n \), \( B_i(t-L) \) for \( \alpha_i, 1 \leq i \leq m \) in (17) and (18), and \( Z_i(t) \) for \( z_i, 1 \leq i \leq m \) in (17), noting that the constraints (9) and (10) yield precisely the constraints in (17).

Taking the expectation of the above conditional expectation,

\[
E \left[ \sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \right] = E \left[ \sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \bigg| \mathcal{F}_{t-L} \right] \geq C_s^*,
\]

which proves the first part of the theorem, equation (19).

To prove the second part of the theorem, we write

\[
E \left[ \sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \bigg| \mathcal{F}_{t-L} \right]
\]

\[
= E \left[ \sum_{i=1}^{m} b_i B_i(t-L) + \sum_{i=1}^{m} b_i D_i(t) + \sum_{j=1}^{n} h_j (I_j(t-L) + R_j(t-L)) - \sum_{i=1}^{m} c_i Z_i(t) \bigg| \mathcal{F}_{t-L} \right]
\]

\[
= \sum_{i=1}^{m} b_i D_i(t) + \sum_{j=1}^{n} h_j (I_j(t-L) + R_j(t-L)) - \sum_{i=1}^{m} a_{ij} B_i(t-L) - E \left[ \sum_{i=1}^{m} c_i (Z_i(t) - B_i(t-L)) \bigg| \mathcal{F}_{t-L} \right]
\]

\[
\geq C_s (I(t-L) + R(t-L) - B(t-L)A)
\]

\[
\geq C_s^*,
\]

where the first inequality comes from substituting \( I_j(t-L) + R_j(t-L) - \sum_{i=1}^{m} a_{ij} B_i(t-L) \) for \( y_j, 1 \leq j \leq n \), in (14) and (15), noting that the constraints (9) and (10), along with (20) yield precisely the constraint in (15). Taking expectations of the above conditional expectation,

\[
E \left[ \sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \right] = E \left[ \sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \bigg| \mathcal{F}_{t-L} \right] \geq C_s^*,
\]

which proves (21).

A similar proof shows that the same lower bounds apply to the periodic-review model presented in Appendix II, in which case the \( \sigma \)-algebra \( \mathcal{F}_t \) is replaced by

\[ F_k = \sigma\{I(0), B(0); d(\kappa), 0 \leq \kappa < k; r(\kappa), 1 - L \leq \kappa < k; z(\kappa), 1 \leq \kappa < k \}. \]
In addition to generally replacing \( t \) by \( k \), the variables \( B(t - L) \) are replaced by \( B(k - L - 1) \), and \( I(t - L) \) are replaced by \( I(k - L - 1) \). With these changes the proof for the periodic-review model proceeds precisely as the above for the continuous-review model.

**Derivation of Equations 22-23**

To prove (22), because \( b_1 \geq b_2 \), it is optimal to satisfy demand for product 1 first and then use the remaining parts to serve product 2. Thus the second-stage recourse problem (15) has the optimal solution:

\[
z_1 = D_1 \land y = D_1 - (D_1 - y)^+ \quad \text{and} \quad z_2 = D_2 \land (y - D_1)^+ = D_2 - (D_2 - (y - D_1)^+)^+.
\]

Inserting the above into (14),

\[
C_y(y) = b_1 E[D_1] + b_2 E[D_2] + h y - (b_1 + h)E[D_1 - (D_1 - y)^+] - (b_2 + h)E[D_2 - (D_2 - (y - D_1)^+)^+]
\]

\[
= b_1 E[(D_1 - y)^+] + b_2 E[(D_2 - (y - D_1)^+)^+] + h E[y - D_1 - D_2 + (D_1 - y)^+ + (D_2 - (y - D_1)^+)^+],
\]

and (22) follows because

\[
y - D_1 - D_2 + (D_1 - y)^+ + (D_2 - (y - D_1)^+)^+ = (y - D_1 - D_2)^+.
\]

To prove (23), because \( b_1 \geq b_2 \), the recourse problem (17) has the optimal solution:

\[
z_1 = (D_1 + \alpha_1) \land y, \quad z_2 = (D_2 + \alpha_2) \land (y - D_1 - \alpha_1)^+.
\]

Inserting the above into (18),

\[
C_y(y, \alpha) = b_1 E[D_1 + \alpha_1] + b_2 E[D_2 + \alpha_2] + h y - (b_1 + h)E[(D_1 + \alpha_1) \land y] - (b_2 + h)E[(D_2 + \alpha_2) \land (y - D_1 - \alpha_1)^+]. \tag{EC-4}
\]

We can assume that \( y \geq \alpha_1 + \alpha_2 \) without loss of optimality. If \( y < \alpha_1 + \alpha_2 \), then increasing \( y \) by \( \Delta y = \alpha_1 + \alpha_2 - y \) in (EC-4) reduces \( C_y(y, \alpha) \) by \( b_1 \Delta y \) on sample paths where \( D_1 + \alpha_1 \geq y + \Delta y \), by \( b_2 \Delta y \) on paths where \( y > D_1 + \alpha_1 \), and by some value in between on all other paths. Thus we can replace \( y \) in (EC-4) with \( y + \alpha_1 + \alpha_2 \) (where \( y \geq 0 \)) and transform the equation into

\[
C_y(y, \alpha) = b_1 E[D_1] + b_2 E[D_2 + \alpha_2] + h(y + \alpha_2) - (b_1 + h)E[D_1 \land (y + \alpha_2)] - (b_2 + h)E[(D_2 + \alpha_2) \land (y - D_1 - \alpha_2)^+] \tag{EC-5}
\]

For any given \( y \), \( C_y(y, \alpha) \) always improves with a higher \( \alpha_2 \) because an increase of \( \alpha_2 \) does not make any difference on sample paths where \( D_1 \leq y + \alpha_2 \), but reduces \( C_y(y, \alpha) \) on other paths. Therefore to reach the minimum of \( C_y(y, \alpha) \), \( \alpha_2 \to \infty \), in which case

\[
E[D_1 \land (y + \alpha_2)] \to E[D_1], \quad E[(D_2 + \alpha_2) \land (y - D_1 + \alpha_2)^+] \to E[(D_1 + D_2) \land y] - E[D_1].
\]

Inserting the above into (EC-5),

\[
\lim_{\alpha_2 \to \infty} C_y(y, \alpha) = b_2 E[D_1 + D_2] + h y - (b_2 + h)E[(D_1 + D_2) \land y],
\]

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which proves (23).

**Proof of Lemma 3.1**

Inserting $z_1(y, \alpha), z_2(y, \alpha)$ into (25),

$$ C^*_a = \min_{y, \alpha \geq 0} \left\{ \sum_{j=0}^{2} h_j y_j + \sum_{i=1}^{2} b_i(\alpha_i + E[D_i]) - c_1 E[(D_1 + \alpha_1) \land y_0 \land y_1] \right. \\
- c_2 E[(D_2 + \alpha_2) \land y_2 \land (y_0 - (D_1 + \alpha_1) \land y_0 \land y_1)] \right\}. \tag{EC-6} $$

If $y_1 > y_0$, then $(D_1 + \alpha_1) \land y_0 \land y_1 = (D_1 + \alpha_1) \land y_0$, and reducing $y_1$ to $y_0$ decreases $C^*_a$ by $h_1(y_1 - y_0)$. Therefore, we can assume $y_1 \leq y_0$ and transform (EC-6) to

$$ C^*_a = \min_{y, \alpha \geq 0} \left\{ \sum_{j=0}^{2} h_j y_j + \sum_{i=1}^{2} b_i(\alpha_i + E[D_i]) - c_1 E[(D_1 + \alpha_1) \land y_1] \right. \\
- c_2 E[(D_2 + \alpha_2) \land y_2 \land (y_0 - (D_1 + \alpha_1) \land y_1)] \right\}. \tag{EC-7} $$

If $y_1 < \alpha_1$, then $(D_1 + \alpha_1) \land y_1 = y_1$ and the cost to be minimized in (EC-7) becomes

$$ \sum_{j=0}^{2} h_j y_j + \sum_{i=1}^{2} b_i(\alpha_i + E[D_i]) - c_1 y_1 - c_2 E[(D_2 + \alpha_2) \land y_2 \land (y_0 - y_1)] \\
= h_0(y_0 - y_1) + h_2 y_2 + \sum_{i=1}^{2} b_i E[D_i] + b_1(\alpha_1 - y_1) + b_2 \alpha_2 - c_2 E[(D_2 + \alpha_2) \land y_2 \land (y_0 - y_1)], $$

which can be reduced by raising $y_1$ to $\alpha_1$ and increasing $y_0$ to keep $y_0 - y_1$ constant. Without loss of optimality, we can thus assume $y_0 \geq y_1 \geq \alpha_1$. Let $y'_0 = y_0 - \alpha_1 \geq 0$, $y'_1 = y_1 - \alpha_1 \geq 0$, and $y'_2 = y_2$, so that

$$(D_1 + \alpha_1) \land y_1 = \alpha_1 + D_1 \land y'_1, \quad y_0 - (D_1 + \alpha_1) \land y_1 = y'_0 - D_1 \land y'_1,$$

and (EC-7) becomes

$$ C^*_a = \min_{y', \alpha_2 \geq 0} \left\{ \sum_{j=0}^{2} h_j y'_j + \sum_{i=1}^{2} b_i E[D_i] + b_2 \alpha_2 - c_1 E[D_1 \land y'_1] \right. \\
- c_2 E[(D_2 + \alpha_2) \land y'_2 \land (y'_0 - D_1 \land y'_1)] \right\}. \tag{EC-8} $$

If $y'_2 > y'_0$, then $y'_2 \land (y'_0 - D_1 \land y'_1) = y'_0 - D_1 \land y'_1$, so letting $y'_2 = y'_0$ reduces the above cost by $h_2(y'_2 - y'_0)$. This leads to $y'_2 \leq y'_0$. Moreover, if $y'_2 < \alpha_2$, then

$$(D_2 + \alpha_2) \land y'_2 \land (y'_0 - D_1 \land y'_1) = y'_2 - (y'_2 - y'_0 + D_1 \land y'_1)^+. $$

By taking out $y'_2$ from inside the last expectation in (EC-8), we transform $C^*_a$ into

$$ h_0(y'_0 - y'_2) + h_1 y'_1 + \sum_{i=1}^{2} b_i E[D_i] + b_2(\alpha_2 - y'_2) - c_1 E[D_1 \land y'_1] + c_2 E[(y'_2 - y'_0 + D_1 \land y'_1)^+], $$
which can be reduced by raising $y_2$ to $\alpha_2$ and $y_0'$ by $\alpha_2 - y_2'$. Therefore, we can assume that $y_0' \geq y_2' \geq \alpha_2$. Let $y_0'' = y_0 - \alpha_2 \geq 0$, $y_1' = y_1$, and $y_2'' = y_2 - \alpha_2 \geq 0$, so

$$(D_2 + \alpha_2) \land (y_0' - D_1 \land y_1') = \alpha_2 + D_2 \land y_2'' \land (y_0'' - D_1 \land y_1''),$$

and thus (EC-8) becomes

$$C_s = \min_{y'' \geq 0} \left\{ \sum_{j=0}^{2} h_{j} y_{j}' + \sum_{i=1}^{2} b_{i} E[D_{i}] - c_{1} E[D_{1} \land y_{1}'] - c_{2} E[D_{2} \land y_{2}'' \land (y_0'' - D_1 \land y_1'')] \right\}. \quad \text{(EC-9)}$$

**Proof of the Statement** that “...finding boundary solutions is equivalent to solving standard newsvendor models.”

We show that the problem of optimizing $C_s(y)$ or $C_s(y')$ at a boundary can be transformed into the following standard form of the newsvendor model:

$$\min_{k \geq 0} \{h_k - cE[D \land k]\}.$$

If $y_1 + y_2 = y_0$, then

$$C_s^* = C_s^* = \sum_{i=1}^{2} b_i E[D_i] + \min_{y \geq 0} \{h_0 (y_1 + y_2) + h_1 y_1 + h_2 y_2 - c_1 E[y_1 \land D_1] - c_2 E[D_2 \land y_2 \land (y_1 + y_2 - y_1 \land D_1)]\}$$

$$= \sum_{i=1}^{2} b_i E[D_i] + \min_{y \geq 0} \{(h_0 + h_1) y_1 - c_1 E[y_1 \land D_1] \} + \min_{y \geq 0} \{(h_0 + h_2) y_2 - c_2 E[y_2 \land D_2] \}.$$

If $y_1 = 0$, then

$$C_s^* = C_s^* = \sum_{i=1}^{2} b_i E[D_i] + \min_{y_0, y \geq 0} \{h_0 y_0 + h_2 y_2 - c_2 E[D_2 \land y_2 \land y_0]\}$$

$$= \sum_{i=1}^{2} b_i E[D_i] + \min_{y \geq 0} \{(h_0 + h_2) y - c_2 E[D_2 \land y] \} \text{ since } y_0 \neq y_2 \text{ is not optimal.}$$

If $y_2 = 0$ and $y_1 \leq y_0$, then $E[D_2 \land y_2 \land (y_0 - y_1 \land D_1)] = 0$, so

$$C_s^* = \sum_{i=1}^{2} b_i E[D_i] + \min_{y_0 \geq y_1} \{h_0 y_0 + h_1 y_1 - c_1 E[y_1 \land D_1]\}$$

$$= \sum_{i=1}^{2} b_i E[D_i] + \min_{y \geq 0} \{(h_0 + h_1) y - c_1 E[y \land D_1]\}.$$

If $y_2 = 0$ and $y_0 \leq y_1$ (which is a possible case of (28)), then

$$D_2 \land y_2 \land (y_0 - D_1 \land y_1) = 0 \land (y_0 - D_1 \land y_1) = (D_1 \land y_0) - (D_1 \land y_1),$$

and therefore,

$$C_s(y_0, y_1, 0) = \sum_{i=1}^{2} b_i E[D_i] + h_1 y_1 - (c_1 - c_2) E[y_1 \land D_1] + h_0 y_0 - c_2 E[D_1 \land y_0].$$
Let
\[ C^* = \sum_{i=1}^{2} b_i E[D_i] + \min_{y_1 \geq 0} \{ h_1 y_1 - (c_1 - c_2) E[y_1 \wedge D_1] \} + \min_{y_0 \geq 0} \{ h_0 y_0 - c_2 E[D_1 \wedge y_0] \}. \]

Then \( C^*_s = C^*_s \wedge C^*_s \).

**Proof of Theorem 3.2**

We first consider a regular solution for (30). When \( y_0 < y_1 + y_2 \),
\[
\begin{align*}
C_s(y_0, y_1, y_2) &= C_s(y_0, y_1, y_2) - C_s(y_0, y_1 - 1, y_2) = h_1 - \bar{F}_1(y_1 - 1) [c_1 - c_2 \bar{F}_2(y_0 - y_1)] = s_1(y_1 | y_0), \\
C_s(y_0, y_1, y_2) &= C_s(y_0, y_1, y_2) - C_s(y_0, y_1, y_2 - 1) = h_2 - c_2 F_1(y_0 - y_2) \bar{F}_2(y_2 - 1) = s_2(y_2 | y_0).
\end{align*}
\]

With our convention that \( F_i(k) = 0 \) for \( k < 0 \), \( s_2(y_2 | y_0) = h_2 \) if \( y_2 > y_0 \). By the definition of regular solution,
\[ s_1(y_1^* | y_0^*) \leq 0 \quad \text{and} \quad s_2(y_2^* | y_0^*) \leq 0. \]

Since, as noted previously, \( s_i(y_i | y_0) \) (\( i = 1, 2 \)) weakly increases in \( y_i \), by the definition of \( Y_i \) in (34),
\[ s_i(Y_i(y_0^*)) \leq 0 < s_i(Y_i(y_0^*) + 1 | y_0^*) \quad i = 1, 2. \]

Because \( 0 < y_1^* \) and \( 0 < y_2^* \) hold for a regular solution, it is necessary that
\[ s_1(1 | y_0^*) \leq 0 \quad \text{and} \quad s_2(1 | y_0^*) \leq 0, \]
which means
\[ h_1 - \bar{F}_1(0) [c_1 - c_2 \bar{F}_2(y_0^* - 1)] \leq 0 \quad \text{and} \quad h_2 - c_2 F_1(y_0^* - 1) \bar{F}_2(0) \leq 0, \]
which is true only if \( Y_0^\text{min} \leq y_0^* \). For \( Y_0^\text{min} < \infty \), it requires
\[ h_i < c_i F_i(0), \quad i = 1, 2. \]

We now derive \( Y_0^\text{max} \). From (EC-10), for \( i = 1, 2, \)
\[ \text{if } c_i \bar{F}_i(y - 1) < h_i \text{ then } s_i(y | y_0) > 0. \]

Since \( \bar{F}_i(y - 1) \) decreases in \( y \) and \( s_i(Y_i(y_0^*)) | y_0 \) \leq 0 (\( i = 1, 2 \)),
\[ Y_i(y_0^*) \leq \min \left\{ k : \bar{F}_i(k - 1) < \frac{h_i}{c_i} \right\}, \quad i = 1, 2, \]
and because \( y_0^* < y_1^* + y_2^* \leq Y_1(y_0^*) + Y_2(y_0^*), y_0^* \leq Y_0^\text{max}. \)

It is possible that \( Y_1(y_0^*) > y_0 \), i.e., \( C_s(y_0^*, y_1, y_2^*) \) always decreases in \( y_1 \) within \([0, y_0]\). In this case, \( C_s \) is maximized at \( y_1^* = y_0^* \), so in general, the optimal solution of (30) is \( (y_0^*, y_0^* \wedge Y_1(y_0^*), Y_2(y_0^*)) \), which can be obtained by a one-dimensional search of \( y_0^* \).

The same proof goes through for optimizing \( C_s(y) \) except that \( y_1^* \) is not bounded by \( y_0^* \), so the optimal solution takes the form \( (y_0^*, Y_1(y_0^*), Y_2(y_0^*)) \).

**Derivation of Equation 35**

For any nonnegative integer random variable \( D \), let \( Y \) be a strictly positive integer, \( Y' \) be a nonnegative integer, and \( Y' < Y \), then
\[ E[D \wedge Y - D \wedge Y'] = \sum_{k=Y'}^{Y-1} \Pr(D > k), \]

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and as a special case, when $Y' = 0$,

$$
E[D \land Y] = \sum_{k=0}^{Y-1} \Pr(D > k).
$$

In $C_s(y)$ and $C_s(x)$, apply the above to $z_1^*(y)$,

$$
E[z_1^*(y)] = E[D_1 \land y_1] = \sum_{k=0}^{y_1-1} \bar{F}_1(k).
$$

Deriving $z_2^*(y)$ is more involved. Given that $y_0 \leq y_1 + y_2$, if $y_1 \leq y_0$, then

$$
E[z_2^*(y)] = E[D_2 \land y_2 \land (y_0 - D_1 \land y_1)]
$$

$$
= E[D_2 \land (y_0 - y_1)] + \sum_{k=0}^{y_1-1} f_1(k) \left[ E[D_2 \land y_2 \land (y_0 - k)] - E[D_2 \land (y_0 - y_1)] \right],
$$

where $f_i(k) = \Pr(D_i = k)$ ($i = 1, 2$). The first part of equation (35) follows because

$$
E[D_2 \land (y_0 - y_1)] = \sum_{k=0}^{y_0-y_1-1} \bar{F}_2(k)
$$

and

$$
\sum_{k=0}^{y_1-1} f_1(k) \left[ E[D_2 \land y_2 \land (y_0 - k)] - E[D_2 \land (y_0 - y_1)] \right]
$$

$$
=F_1(y_0 - y_2 - 1) \left[ E[D_2 \land y_2 \land y_0 - D_2 \land (y_0 - y_1)] \right] + \sum_{k=0}^{y_1-1} f_1(k) \left[ E[D_2 \land (y_0 - k) - D_2 \land (y_0 - y_1)] \right]
$$

$$
=F_1(y_0 - y_2 - 1) \sum_{k=0}^{y_2-1} \bar{F}_2(k) + \sum_{k=0}^{y_1-1} f_1(k) \sum_{k'=0}^{y_0-k-1} \bar{F}_2(k')
$$

$$
=F_1(y_0 - y_2 - 1) \sum_{k=0}^{y_2-1} \bar{F}_2(k) + \sum_{k=0}^{y_1-1} f_1(k) \sum_{k'=0}^{y_0-k-1} \bar{F}_2(k')
$$

$$
= \sum_{k=0}^{y_2-1} F_1(y_0 - k - 1) \bar{F}_2(k).
$$

If $y_1 > y_0$ (which only applies to $C_s(x)$),

$$
E[z_2^*(y)] = E[D_2 \land y_2 \land (y_0 - D_1 \land y_1)]
$$

$$
= F_1(y_0 - y_2 - 1) E[y_2 \land D_2] + \sum_{k=0}^{y_1-1} f_1(k) E[D_2 \land (y_0 - k)] + E[(y_0 - D_1 \land y_1)^-]
$$

$$
= F_1(y_0 - y_2 - 1) \sum_{k=0}^{y_2-1} \bar{F}_2(k) + \sum_{k=0}^{y_1-1} f_1(k) \sum_{k'=0}^{y_0-k-1} \bar{F}_2(k') - E[D_1 \land y_1 - D_1 \land y_0]
$$

$$
= \sum_{k=0}^{y_2-1} F_1(y_0 - k - 1) \bar{F}_2(k) - \sum_{k=0}^{y_1-1} \bar{F}_1(k),
$$

7
which leads to the second part of (35).

**Proof of Lemma 3.3**

When \( y_0 > y_1 + y_2 \), then by (36) and (37)

\[
P_0^P(t) = y_0 - y_1 - y_2 + P_i^P(t) + P_2^P(t) > 0 \quad \text{for all } t \geq 0.
\]  

(EC-11)

Combining (37) and (38) we obtain

\[
B_i^P(t) = D_i(t) - y_i + P_i^P(t), \quad i = 1, 2.
\]  

(EC-12)

From (39) and (EC-11) we have

\[
P_i^P(t)B_i^P(t) = 0, \quad i = 1, 2.
\]  

(EC-13)

Equation (40) is immediate from (EC-12) and (EC-13), along with the condition that \( P_i^P(t), B_i^P(t) \geq 0 \) (\( i = 1, 2 \)).

We now prove (41). By canceling out \( B_i^P(t - L) - Z_i^P(t) \) (\( i = 1, 2 \)) in (36)-(38),

\[
D_1(t) + D_2(t) - y_0 = B_0^P(t) + B_1^P(t) - P_0^P(t),
\]

\[
D_i(t) - y_i = B_i^P(t) - P_i^P(t), \quad i = 1, 2.
\]

Because \( P_i^P(t) \geq 0 \) (\( j = 0, 1, 2 \)),

\[
B_1^P(t) + B_2^P(t) \geq (D_1(t) + D_2(t) - y_0)^+ \lor (D_1(t) - y_1)^+ \lor (D_2(t) - y_2)^+,
\]

so we can prove (41) by showing that

\[
B_1^P(t) + B_2^P(t) \leq (D_1(t) + D_2(t) - y_0)^+ \lor (D_1(t) - y_1)^+ \lor (D_2(t) - y_2)^+.
\]  

(EC-14)

When \( y_0 \leq y_1 + y_2 \), equations (36) and (37) imply that

\[
P_0^P(t) = P_i^P(t) + P_2^P(t) + y_0 - y_1 - y_2 \leq P_i^P(t) + P_2^P(t).
\]

Hence \( P_i^P(t) = 0 \) if \( P_1^P(t) = P_2^P(t) = 0 \). In order for the defining property of a myopic policy

\[
[I_i^P(t) \land P_0^P(t)]B_i^P(t) = 0, \quad i = 1, 2,
\]

to hold, at least one of the following must be true:

- \( B_1^P(t) = B_2^P(t) = 0 \), in which case (EC-14) is obviously satisfied.
- \( I_i^P(t) = 0 \) and \( B_2^P(t) = 0 \) (\( i = 1, 2 \)), in which case (EC-14) is true because

\[
B_1^P(t) + B_2^P(t) = B_1^P(t) = D_1(t) - y_1.
\]

- \( I_0^P(t) = 0 \), in which case (EC-14) is also true because

\[
B_1^P(t) + B_2^P(t) = D_1(t) + D_2(t) - y_0.
\]

To prove the second equality in (41),

\[
D_1 - z_1^*(y) + D_2 - z_2^*(y) = D_1 - D_1 \lor y_1 + D_2 - D_2 \lor y_2 \lor (y_0 - D_1 \lor y_1)
\]

\[
= (D_1 - y_1)^+ + (D_2 - y_2 \lor (y_0 - D_1 \lor y_1))^+.
\]  

(EC-15)
• If $D_2 < y_2 \land (y_0 - D_1 \land y_1)$, then $(D_2 - y_2)^+ = 0$, and (EC-15) shows that

$$D_1 - z_1^*(y) + D_2 - z_2^*(y) = (D_1 - y_1)^+,$$

and

$$(D_1 + D_2 - y_0)^+ \leq (D_1 + y_0 - D_1 \land y_1 - y_0)^+ = (D_1 - y_1)^+. $$

• If $D_2 \geq y_2 \land (y_0 - D_1 \land y_1)$ and $D_1 < y_1$, then $(D_1 - y_1)^+ = 0$, and by (EC-15),

$$D_1 - z_1^*(y) + D_2 - z_2^*(y) = (D_2 - y_2 \land (y_0 - D_1))^+ = (D_1 + D_2 - y_0)^+ \lor (D_2 - y_2)^+.$$ 

• If $D_2 \geq y_2 \land (y_0 - D_1 \land y_1)$ and $D_1 \geq y_1$, then by (EC-15),

$$D_1 - z_1^*(y) + D_2 - z_2^*(y) \geq (D_1 - y_1)^+.$$ 

Because $y_0 \leq y_1 + y_2$, $D_2 \geq y_0 - y_1$. Thus $D_1 + D_2 - y_0 \geq 0$ and

$$D_1 - z_1^*(y) + D_2 - z_2^*(y) = D_1 - y_1 + D_2 - y_2 \land (y_0 - y_1) = (D_1 + D_2 - y_0)^+ \geq (D_2 - y_2)^+.$$ 

This proves that the equality holds in all cases.

**Proof of Theorem 3.4**

The following proof is based on the continuous review model but can be directly extended to the periodic review model. From (36)-(38),

$$I_0^P(t) = y_0 + B_1^P(t) + B_2^P(t) - D_1(t) - D_2(t), \quad I_i^P(t) = y_i + B_i^P(t) - D_i(t), \quad (i = 1, 2),$$

so in (8),

$$\sum_{i=1}^2 b_i E[B_i^P(t)] + \sum_{j=0}^2 h_j E[I_j^P(t)] = \sum_{j=0}^2 h_j y_j - \sum_{i=1}^2 (h_0 + h_i) E[D_i(t)] + \sum_{i=1}^2 c_i E[B_i^P(t)]. \tag{EC-16}$$

In (30),

$$\sum_{j=0}^2 h_j y_j + \sum_{i=1}^2 b_i E[D_i] - \sum_{i=1}^2 c_i E[z_i^*(y)] = \sum_{j=0}^2 h_j y_j - \sum_{i=1}^2 (h_0 + h_i) E[D_i] + \sum_{i=1}^2 c_i E[D_i - z_i^*(y)]. \tag{EC-17}$$

The right hand sides of (EC-16) and (EC-17) are equal when $c_1 = c_2$ and $D_i(t) = D_i$ because by Lemma 3.3, $B_i^P(t) + B_2^P(t) = D_1 + D_2 - z_1^*(y) - z_2^*(y)$. This completes the proof of (44).

To prove (45), note that, by Lemma 3.3 it is sufficient to show that

$$B_i^P(t) \geq B_i^{PBC}(t) \tag{EC-18}$$

for any $p \in P_m$ and $t \geq 0$. Suppose that (EC-18) does not hold, and let $\tau$ denote the first time that (EC-18) is violated. Then $\Delta Z_i^P(\tau) > \Delta Z_i^{PBC}(\tau) + \delta$, where $\delta \equiv B_i^P(\tau^-) - B_i^{PBC}(\tau^-) \geq 0$. By (42) we have

$$\Delta Z_i^P(\tau) > \min\{B_1^{PBC}(\tau^-) + \Delta D_1(\tau), \quad I_0^{PBC}(\tau^-) + \Delta R_0(\tau), \quad I_1^{PBC}(\tau^-) + \Delta R_1(\tau)\} + \delta.$$

There are three possible cases:
1. $\Delta Z^p_1(\tau) > B^P_1(t) + \Delta D_1(\tau) + \delta = B^P_1(\tau) + \Delta D_1(\tau)$.

Using (6) with $t = \tau$ it is clear that in this case $B^P_1(\tau) < 0$, so $p$ is not feasible.

2. $\Delta Z^p_1(\tau) > I^P_0(\tau) - \Delta R_0(\tau) + \delta$.

Note that, by Lemma 3.3, $I^P_0(t) = I^P_0(t)$ for all $t \geq 0$. Also, $\Delta Z_2(t) \geq 0$ for all $t \geq 0$. Thus, using (7) with $j = 0$ and $t = \tau$, we obtain

$$I^P_0(\tau) < I^P_0(\tau) - I^P_0(\tau) - \Delta Z_2(t) - \delta \leq 0,$$

so $p$ is not feasible.

3. $\Delta Z^p_1(\tau) > I^P_0(\tau) + \Delta R_0(\tau) + \delta$.

We have $I^P_1(t) - I^P_0(t) = B^P_1(t) - B^P_0(t)$ for all $t \geq 0$, so that $I^P_1(t) - I^P_0(t) = -\delta$. Using (7) with $j = 1$ and $t = \tau$ we obtain

$$I^P_1(\tau) < I^P_1(\tau) - I^P_0(\tau) - \delta = -2\delta,$$

so, again, $p$ is not feasible.

Thus (EC-18) always holds.

**Proof of Theorem 3.5**

When $c_1 = c_2$, it is immediate from (44) that $C_s(\mathbf{y}^*) = C^{PBC}(\mathbf{y}^*)$, so we only need to show that $C_s(\mathbf{y}^*) = C^*_s$. This is proved by showing that if $\mathbf{y}^{**}$ is the optimal solution of (31), then $y^{**}_1 \leq y^{*}_0$, so $\mathbf{y}^{**}$ is a feasible solution of (30).

Suppose $y^{**}_0 < y^{*}_1$, then (31) applies. For the discrete demand formulation,

$$C_s(y^{**}_0, y^{**}_1, y^{**}_2) - C_s(y^{*}_0, y^{*}_1, y^{*}_2) = -h_1 + (c_1 - c_2)F_1(y^{*}_1 - 1).$$

Because $y^{*}_1$ is optimal, the above has to be nonnegative. When $c_1 = c_2$, this means $-1 \geq 0$, which does not hold, so $y^{**}_0 < y^{*}_1$ cannot be true.

The same argument applies to the continuous demand formulation if we use $\partial C_s/\partial y_1$, evaluated at $y^{**}$, to replace $C_s(y^{**}_0, y^{**}_1, y^{**}_2) - C_s(y^{*}_0, y^{*}_1, y^{*}_2)$ in the above.

When $y_1 + y_2 = y_0$, $y_0 - D_1 \wedge y_1 \geq y_2$, so in (30),

$$C_s(y_0, y_1, y_2) = (h_0 + h_1)y_1 + (h_0 + h_2)y_2 + \sum_{i=1}^{2} b_i E[D_i] - c_1 E[D_1 \wedge y_1] - c_2 E[D_2 \wedge y_2]$$

$$= \sum_{i=1}^{2} (h_i + h_0) E[(y_i - D_i)^+] + \sum_{i=1}^{2} b_i E[(D_i - y_i)^+]$$

(19)

where the second equality makes use of $D_1 \wedge y_1 = y_1 - (D_i - y_i)^+ = D_i - (D_i - y_i)^+ (i = 1, 2)$.

In the inventory system, equations (41), (EC-12), along with conditions that both inventory and backlog levels are nonnegative, imply that

$$B^P_i(t) = (D_i(t) - y_i^*)^+, \quad I^P_i(t) = (y_i^* - D_i(t))^+, \quad (i = 1, 2),$$

and $I^P_0(t) = (y_1^* - D_1(t))^+ + (y_2^* - D_2(t))^+$,

which together with (8) and (EC-19), imply that

$$C_s(\mathbf{y}^*) = C^{PBC}(\mathbf{y}^*).$$
It remains to be shown that \( C_s^*(y^*) = C_s^* \). Following the same argument as above, we prove that there exists at least one optimal solution \( y^{**} \) for (31) such that \( y_1^{**} \leq y_0^{**} \), so \( y^{**} \) is a feasible solution of (30).

We first consider the discrete demand formulation. Because \( y_0^* \) and \( y_1^* \) are optimal for (30),
\[
\begin{align*}
C_s(y_0^* - 1, y_1^*, y_2^*) - C_s(y_0^*, y_1^*, y_2^*) &= -h_0 + c_2 \Pr\{D_1 \geq y_1^*, D_2 \geq y_2^*\} \\
C_s(y_0^*, y_1^* + 1, y_2^*) - C_s(y_0^*, y_1^*, y_2^*) &= h_1 - c_1 \bar{F}_1(y_1^*) + c_2 \Pr\{D_1 > y_1^*, D_2 \geq y_2^*\} \geq 0,
\end{align*}
\]
which implies that
\[
\bar{F}_1(y_1^*) \leq \frac{h_1}{c_1 - c_2} \quad \text{and} \quad \bar{F}_1(y_1^* - 1) \geq \frac{h_0}{c_2} \quad \text{(EC-20)}
\]

The optimal solution of (31) may not be unique. We choose \( y^{**} \) such that \( y_1^{**} - y_0^{**} \) is the minimum of all optimal solutions, which ensures the following strict inequalities
\[
\begin{align*}
\bar{C}_s(y_0^{**} + 1, y_1^{**}, y_2^{**}) - \bar{C}_s(y_0^{**}, y_1^{**}, y_2^{**}) &> 0, \\
\bar{C}_s(y_0^{**}, y_1^{**} - 1, y_2^{**}) - \bar{C}_s(y_0^{**}, y_1^{**}, y_2^{**}) &> 0.
\end{align*}
\]

Using (31) to expand the left-hand sides of the above,
\[
\begin{align*}
h_0 - c_2 \Pr\{D_2 \wedge y_2^{**} > y_0^{**} - D_1 \geq 0\} - c_2 \bar{F}_1(y_0^{**}) &> 0, \\
-h_1 + (c_1 - c_2) \bar{F}_1(y_1^{**} - 1) &> 0,
\end{align*}
\]
which implies that
\[
\bar{F}_1(y_0^{**}) < \frac{h_0}{c_2} \quad \text{and} \quad \bar{F}_1(y_1^{**} - 1) > \frac{h_1}{c_1 - c_2}. \quad \text{(EC-21)}
\]

Following from (EC-20) and (EC-21),
\[
y_1^* - 1 < y_0^{**} \quad \text{and} \quad y_1^{**} - 1 < y_1^*.
\]

Since \( y_0^{**} \), \( y_1^{**} \), and \( y_1^* \) are all integers, the above holds only if \( y_0^{**} \geq y_1^{**} \).

A similar argument carries over to the continuous demand formulation. By using partial derivatives in place of differences of \( C_s \) and \( \bar{C}_s \), we arrive at following analogous inequalities to (EC-20) and (EC-21),
\[
\frac{h_0}{c_2} \leq F_1(y_0^*) \leq \frac{h_1}{c_1 - c_2}, \quad F_1(y_0^{**}) \leq \frac{h_0}{c_2}, \quad \text{and} \quad F_1(y_1^{**}) \geq \frac{h_1}{c_1 - c_2},
\]
so \( y_1^{**} \leq y_1^* \leq y_0^{**} \).

II. Periodic-Review Formulation

In the following, we develop a periodic-review (discrete time) formulation that exactly parallels the continuous-review model presented in Section 2. With slight modification to definitions, all results in the paper carry over.

There are \( m \) products and \( n \) components and \( a_{ij} \) is the amount of component \( j \) (1 \( \leq j \leq n \)) needed to assemble product \( i \) (1 \( \leq i \leq m \)). Let \( k \geq 1 \) be the index of review periods. Let \( h_j \) denote the inventory holding cost per-period of component \( j \) (1 \( \leq j \leq n \)) and \( b_i \) denote the backlog cost per-period of product \( i \) (1 \( \leq i \leq m \)). At the beginning of period \( k \), an order of \( \mathbf{r}(k) = \{r_1(k),...,r_n(k)\} \) for components 1 \( \leq j \leq n \) is placed based on a replenishment policy \( \gamma \). All components have a common, deterministic replenishment lead time \( L \). Both product demands in this period, \( d_i(k), 1 \leq i \leq m \), and component orders placed \( L \) periods earlier, \( r_j(k-L), 1 \leq j \leq n \),
arrive by the end of period $k$. For $k \geq 1$, let $d(k) \equiv (d_1(k), \ldots, d_m(k))$. We assume that $\{d(k), k \geq 1\}$ is an i.i.d. sequence. We also assume that $E[d_i(1)] < \infty$ for all $1 \leq i \leq m$. Under an allocation policy $p$, an amount $z_i(k)$ of product $i$ demands are served, giving rise to end of period backlog levels

$$B_i(k) = B_i(k-1) + d_i(k) - z_i(k), \quad 1 \leq i \leq m,$$

and end of period on-hand inventory levels

$$I_j(k) = I_j(k-1) + r_j(k - L) - \sum_{i=1}^{m} a_{ij} z_i(k), \quad 1 \leq j \leq n.$$

As for initial conditions, $I_j(0), 1 \leq j \leq n$, are the inventory levels and $B_i(0), 1 \leq i \leq m$, are the backlog levels at the beginning of the first period, and $r_j(1 - L), \ldots, r_j(0)$ are the initial orders in the pipeline that will arrive by the end of periods $1, \ldots, L$, respectively.

Adding up each equation from $k - L$ to $k$ and canceling out redundant terms, we get the following analogous conditions to (1) and (2) of the continuous-review model:

$$B_i(k) = B_i(k - L - 1) + D_i(k) - Z_i(k), \quad 1 \leq i \leq m, \quad \text{(EC-22)}$$

$$I_j(k) = I_j(k - L - 1) + R_j(k - L) - \sum_{i=1}^{m} a_{ij} Z_i(k), \quad 1 \leq j \leq n, \quad \text{(EC-23)}$$

where $R_j(k) \equiv r_j(k - L) + \ldots + r_j(k)$ is the amount of component $j$ ordered from period $k - L$ to $k$, and $D_i(k) \equiv d_i(k - L) + \ldots + d_i(k)$ and $Z_i(k) \equiv z_i(k - L) + \ldots + z_i(k)$ are respectively the amounts of product $i$ demand arrived and served from period $k - L$ to period $k$. The quantity $D_i(k)$ is also referred to as the lead-time demand.

We seek to minimize the long-run average expected cost, which for the periodic-review model is defined as

$$C^{\gamma,p} \equiv \limsup_{K \to \infty} \frac{1}{K} E \left[ \sum_{k=1}^{K} \left\{ \sum_{i=1}^{m} b_i B_i(k) + \sum_{j=1}^{n} h_j I_j(k) \right\} \right].$$

Parallel to the continuous-review formulation, a feasible policy for the periodic-review model must satisfy conditions that

1. For $k \geq 1$, $B_i(k) \geq 0, 1 \leq i \leq m$. From (EC-22), this implies that

$$Z_i(k) \leq B_i(k - L - 1) + D_i(k), \quad 1 \leq i \leq m.$$

2. For $k \geq 1$, $I_j(k) \geq 0, 1 \leq j \leq n$. From (EC-23), this implies that

$$\sum_{i=1}^{m} a_{ij} Z_i(k) \leq I_j(k - L - 1) + R_j(k - L), \quad 1 \leq j \leq n.$$

3. For all $k \geq 1$, $r(k)$ is chosen using only the information $I(0), B(0), \{d(l), 1 \leq l < k\}, \{r(l), 1 - L \leq l < k\}$, and $z(l), 1 \leq l < k$, while $z(k)$ is chosen using only the information $I(0), B(0), \{d(l), 1 \leq l \leq k\}, \{r(l), 1 - L \leq l \leq k\}$, and $\{z(l), 1 \leq l \leq k\}$. 

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III. Solving the Continuous Stochastic Program for the W System

At the end of Section 3.2, we state that “...because the continuous formulation yields smooth objective functions, both $C_s(y)$ and $C_s(y)$ can be reduced to convex functions of $y_0$. Consequently, we can find $y_0^*$ by a one-dimensional bisection search instead of enumerating $y_0$ over $[Y_0^{min}, Y_0^{max}]$.” We now provide more details about this statement.

Analogous to $s_i(y_i|y_0)$ (i = 1, 2) in (EC-10) for the discrete case, let

$$
S_1(y_1|y_0) = \frac{\partial C_s(y)}{\partial y_1} = h_1 - \bar{F}_1(y_1)[c_1 - c_2 \bar{F}_2(y_0 - y_1)],
$$
$$
S_2(y_2|y_0) = \frac{\partial C_s(y)}{\partial y_2} = h_2 - c_2 F_1(y_0 - y_2) F_2(y_2),
$$
in the continuous model. To minimize $C_s(y)$ at $y_1 > 0$, $y_2 > 0$ and $y_0 < y_1 + y_2$, it is necessary that

$$
S_i(y_i|y_0) = 0, \quad i = 1, 2. \tag{EC-24}
$$

Because $S_i(y_i|y_0)$ (i = 1, 2) increase in $y_i$, the solution exists only if $S_i(0|y_0) < 0$, i.e.,

$$
y_0 \geq Y_0^{min} \equiv \bar{F}_2^{-1}\left(1 - \frac{h_1}{c_2} \right) \lor F_1^{-1}\left(\frac{h_2}{c_2}\right).
$$

When (EC-24) is true, $c_1 \bar{F}_1(y_i) \geq h_i$ (i = 1, 2), so $y_0 < y_1 + y_2$ makes it necessary that

$$
y_0 < Y_0^{max} \equiv \bar{F}_1^{-1}\left(\frac{h_1}{c_1}\right) + \bar{F}_2^{-1}\left(\frac{h_2}{c_2}\right).
$$

It is easy to verify that when $y_0 \in [Y_0^{min}, Y_0^{max}]$,

$$
S_1(0|y_0) < 0 < S_1(\infty|y_0), \quad \text{and} \quad S_2(0|y_0) < 0 < S_2(y_0|y_0),
$$

and because under the assumption that $D_i$ have strictly positive densities, $S_i(y_i|y_0)$ strictly increase in $y_i$ (i = 1, 2), (EC-24) has a unique solution $Y_1(y_0) \in (0, \infty)$ and $Y_2(y_0) \in (0, y_0)$.

We first consider cases where

$$
Y_0^{min} \geq \bar{Y}_0 \equiv \bar{F}_2^{-1}\left(1 - \frac{h_1}{c_1 - c_2} \right), \tag{EC-25}
$$

implying that $S_1(Y_0^{min}|Y_0^{min}) \geq 0$. Because $S_1(0|y_0)$ increases in $y_0$, (EC-25) means that for all $y_0 \in [Y_0^{min}, Y_0^{max}]$, $S_1(y_1|y_0) \geq 0$ at $y_1 = y_0$, and thus $Y_1(0) \leq y_0$.

At $(y_0, Y_1(y_0), Y_2(y_0))$,

$$
\frac{dC_s}{dy_0} = h_0 - c_2[\Pr(D_1 \geq Y_1, D_2 \geq y_0 - Y_1) + \Pr(y_0 - Y_2 \leq D_1 < Y_1, D_1 + D_2 \geq y_0)]
= h_0 - c_2 \left[\bar{F}_1(Y_1) \bar{F}_2(y_0 - Y_1) + \int_{y_0 - Y_2}^{Y_1} f_1(x) \bar{F}_2(y_0 - x) dx\right],
$$

and $dC_s/dy_0 = 0$ is the first-order necessary condition for the regular solution. Applying implicit differentiation to $S_1(y_1|y_0) = 0$ and $S_2(y_2|y_0) = 0$,

$$
\frac{dY_1}{dy_0} = \frac{c_2 \bar{F}_1(Y_1) f_2(y_0 - Y_1)}{f_1(Y_1)[c_1 - c_2 \bar{F}_2(y_0 - Y_1)] + c_2 \bar{F}_1(Y_1) f_2(y_0 - Y_1)} \in [0, 1], \tag{EC-26}
$$
$$
\frac{dY_2}{dy_0} = \frac{f_1(y_0 - Y_2) \bar{F}_2(Y_2)}{\bar{F}_1(y_0 - Y_2) \bar{F}_2(Y_2) + f_1(y_0 - Y_2) \bar{F}_2(Y_2)} \in [0, 1]. \tag{EC-27}
$$
It follows that \( \frac{dC_s}{dy_0} \) monotonically increase in \( y_0 \) because

\[
\frac{d^2C_s(y_0)}{dy_0^2} = c_2 \bar{f}_1(Y_1) f_2(y_0 - Y_1) \left( 1 - \frac{dY_1}{dy_0} \right) + c_2 f_1(y_0) \bar{F}_2(y_0 - Y_1) \left( 1 - \frac{dY_1}{dy_0} \right) + c_2 f_1(y_0 - Y_2) \bar{F}_2(y_2) \left( 1 - \frac{dY_2}{dy_0} \right) + c_2 \int_{y_0 - Y_2}^{Y_1} f_1(x) f_2(y_0 - x) \, dx
\]

\[
\geq 0 \quad \text{because} \quad \frac{dY_1}{dy_0}, \frac{dY_2}{dy_0} \leq 1.
\]

Therefore, \( \frac{dC_s}{dy_0} = 0 \) can be solved by a simple bisection search. The root is the regular solution if it satisfies \( y_0 < Y_1(y_0) + Y_2(y_0) \). Otherwise, the constraint and the first-order condition cannot be satisfied simultaneously, indicating the regular solution does not exist.

For cases where \( Y_0^{\text{min}} < \bar{Y}_0 \), we search for the regular solution in \( [Y_0^{\text{min}}, \bar{Y}_0] \) and \( [\bar{Y}_0, Y_0^{\text{max}}] \) separately. The same procedure as the one above applies to the search in the second interval while a minor change is needed for the search in the first interval. For \( y_0 \in [Y_0^{\text{min}}, \bar{Y}_0] \), \( S_1(y_1|y_0) < 0 \) at \( y_1 = y_0 \), so \( Y_1(y_0) > y_0 \), and given \( y_0, y_1 = y_0 \) and \( y_2 = Y_2(y_0) \) minimize \( C_s(y) \) under the constraint \( y_1 \leq y_0 \). Hence

\[
C_s(y_0) = \sum_{i=1}^{2} b_i E[D_i] + (h_0 + h_1) y_0 + h_2 Y_2(y_0) - c_1 E[y_0 \land D_1] - c_2 E[D_2 \land Y_2(y_0) \land (y_0 - y_0 \land D_1)].
\]

We prove that \( C_s(y_0) \) is convex over \( [Y_0^{\text{min}}, \bar{Y}_0] \) so a simple bisection search also suffices. Because \( \frac{dC_s}{dy_0} = 0 \) at \( y_2 = Y_2(y_0) \),

\[
\frac{dC_s}{dy_0} = h_1 + h_0 - c_1 \bar{F}_1(y_0) - c_2 \text{Pr}(y_0 - Y_2(y_0) \leq D_1 \leq y_0 \leq D_1 + D_2)
\]

\[
= h_1 + h_0 - c_1 \bar{F}_1(y_0) - c_2 \int_{y_0 - Y_2(y_0)}^{y_0} f_1(x) \bar{F}_2(y_0 - x) \, dx,
\]

\[
\frac{d^2C_s(y_0)}{dy_0^2} = (c_1 - c_2) f_1(y_0) + c_2 f_1(y_0 - Y_2(y_0)) \bar{F}_2(y_2(y_0)) \left( 1 - \frac{dY_2}{dy_0} \right) + \int_{y_0 - Y_2(y_0)}^{y_0} f_1(x) f_2(y_0 - x) \, dx
\]

\[
\geq 0 \quad \text{because} \quad \frac{dY_2}{dy_0} \leq 1 \quad \text{by (EC-26)}.
\]

Note for \( Y_0^{\text{min}} < y_0 \leq \bar{Y}_0, y_1 = y_0, y_2 > 0 \), so the constraint \( y_0 < y_1 + y_2 \) is always satisfied here.

The same approach applies to the optimization of \( C_s(y) \) except that there is no need to divide the search into two regions when \( Y_0^{\text{min}} < \bar{Y}_0 \) because the constraint \( y_1 \leq y_0 \) no longer applies.