OPERATIONS RESEARCH
DOI 10.1287/opre.1090.0772ec

e-companion

ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—“A Stochastic Programming Based Inventory Policy for Assemble-to-Order Systems with Application to the W Model” by Mustafa K. Doğru, Martin I. Reiman, and Qiong Wang, *Operations Research*, DOI 10.1287/opre.1090.0772.

Electronic Companion

This e-companion is composed of three parts. Part I proves statements, equations, lemmas, and theorems in the paper. Part II presents the periodic-review (discrete time) formulation of our model as a supplement to the continuous-review formulation in the paper. Part III discusses a simple algorithm for solving our SP as a continuous optimization problem. Equations in the companion are referred by "EC-xx" and equations in the paper are referred by their numbers without the prefix "EC".

I. Proofs of Theorems

Proof of Theorem 2.1

We prove that, for any $t \geq L$,

$$\underline{C}_s^* \leq \mathsf{E}\left[\sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t)\right]. \tag{EC-1}$$

By the definition of $C^{\gamma,p}$ in (8), this yields the claimed result. By (1) and (2),

$$\begin{aligned}
&\sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \qquad \text{(EC-2)}\\
&= \sum_{i=1}^{m} b_i\big(B_i(t-L) + D_i(t) - Z_i(t)\big) + \sum_{j=1}^{n} h_j\Big(I_j(t-L) + R_j(t-L) - \sum_{i=1}^{m} a_{ij} Z_i(t)\Big)\\
&= \sum_{i=1}^{m} b_i\big(D_i(t) + B_i(t-L)\big) + \sum_{j=1}^{n} h_j\big(I_j(t-L) + R_j(t-L)\big) - \sum_{i=1}^{m} c_i Z_i(t).
\end{aligned}$$

We need to introduce additional notation for the proof. In particular, we need to introduce the σ-algebra

$$\mathcal{F}_t = \sigma\{\mathbf{I}(0^-), \mathbf{B}(0^-);\ \mathcal{R}(s), -L \leq s \leq t;\ \mathcal{D}(s), \mathcal{Z}(s), 0 \leq s \leq t\},$$

which represents the information available after any decisions at time t have been made. It follows that

$$\mathsf{E}\big[B_i(t-L)|\mathcal{F}_{t-L}\big] = B_i(t-L), \quad 1 \leq i \leq m. \tag{EC-3}$$

Similarly,

$$\mathsf{E}\big[I_j(t-L)|\mathcal{F}_{t-L}\big] = I_j(t-L) \ \text{ and } \ \mathsf{E}\big[R_j(t-L)|\mathcal{F}_{t-L}\big] = R_j(t-L), \ \ 1 \leq j \leq n.$$

In addition, since $\{\mathcal{D}(t), t \geq 0\}$ is a compound Poisson process, $\mathbf{D}(t)$ is independent of $\mathcal{F}_{t-L}$ so that, conditioned on $\mathcal{F}_{t-L}$, $\mathbf{D}(t)$ is equal in distribution to $\mathbf{D}$, where $\mathbf{D}$ is defined as a random variable that has the same distribution as $\mathbf{D}(t)$ and is independent of $\{\mathcal{D}(t), t \geq 0\}$. Taking the conditional expectation of (EC-2) with respect to $\mathcal{F}_{t-L}$ and imposing the feasibility conditions (9) and (10) on $\mathbf{Z}(t)$, we thus obtain

$$\begin{aligned}
&\mathsf{E}\left[\sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \big| \mathcal{F}_{t-L}\right] \\
&= \mathsf{E}\left[\sum_{i=1}^{m} b_i B_i(t-L) + \sum_{i=1}^{m} b_i D_i(t) + \sum_{j=1}^{n} h_j \big(I_j(t-L) + R_j(t-L)\big) - \sum_{i=1}^{m} c_i Z_i(t) \big| \mathcal{F}_{t-L}\right] \\
&= \sum_{i=1}^{m} b_i B_i(t-L) + \sum_{j=1}^{n} h_j \big(I_j(t-L) + R_j(t-L)\big) + \mathsf{E}\left[\sum_{i=1}^{m} b_i D_i\right] - \mathsf{E}\left[\sum_{i=1}^{m} c_i Z_i(t) \big| \mathcal{F}_{t-L}\right] \\
&\geq \underline{C}_s(\mathbf{I}(t-L) + \mathbf{R}(t-L), \mathbf{B}(t-L)) \\
&\geq \underline{C}_s^*,
\end{aligned}$$

where the first inequality comes from substituting $I_j(t-L) + R_j(t-L)$ for $y_j, 1 \leq j \leq n$, $B_i(t-L)$ for $\alpha_i, 1 \leq i \leq m$ in (17) and (18), and $Z_i(t)$ for $z_i, 1 \leq i \leq m$ in (17), noting that the constraints (9) and (10) yield precisely the constraints in (17).

Taking the expectation of the above conditional expectation,

$$\mathsf{E}\left[\sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t)\right] = \mathsf{E}\left[\mathsf{E}\left[\sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \big| \mathcal{F}_{t-L}\right]\right] \geq \underline{C}_s^*,$$

which proves the first part of the theorem, equation (19).

To prove the second part of the theorem, we write

$$\begin{aligned}
&\mathsf{E}\left[\sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \big| \mathcal{F}_{t-L}\right] \\
&= \mathsf{E}\left[\sum_{i=1}^{m} b_i B_i(t-L) + \sum_{i=1}^{m} b_i D_i(t) + \sum_{j=1}^{n} h_j \big(I_j(t-L) + R_j(t-L)\big) - \sum_{i=1}^{m} c_i Z_i(t) \big| \mathcal{F}_{t-L}\right] \\
&= \mathsf{E}\left[\sum_{i=1}^{m} b_i D_i\right] + \sum_{j=1}^{n} h_j \Big(I_j(t-L) + R_j(t-L) - \sum_{i=1}^{m} a_{ij} B_i(t-L)\Big) - \mathsf{E}\left[\sum_{i=1}^{m} c_i (Z_i(t) - B_i(t-L)) \big| \mathcal{F}_{t-L}\right] \\
&\geq C_s(\mathbf{I}(t-L) + \mathbf{R}(t-L) - \mathbf{B}(t-L)A) \\
&\geq C_s^*,
\end{aligned}$$

where the first inequality comes from substituting $I_j(t-L) + R_j(t-L) - \sum_{i=1}^{m} a_{ij} B_i(t-L)$ for $y_j, 1 \leq j \leq n$, in (14) and (15), noting that the constraints (9) and (10), along with (20) yield precisely the constraint in (15). Taking expectations of the above conditional expectation,

$$\mathsf{E}\left[\sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t)\right] = \mathsf{E}\left[\mathsf{E}\left[\sum_{i=1}^{m} b_i B_i(t) + \sum_{j=1}^{n} h_j I_j(t) \big| \mathcal{F}_{t-L}\right]\right] \geq C_s^*,$$

which proves (21).

A similar proof shows that the same lower bounds apply to the periodic-review model presented in Appendix II, in which case the σ-algebra $\mathcal{F}_t$ is replaced by

$$\mathcal{F}_k = \sigma\{\mathbf{I}(0), \mathbf{B}(0);\ \mathbf{d}(\kappa), 0 \leq \kappa < k;\ \mathbf{r}(\kappa), 1-L \leq \kappa \leq k;\ \mathbf{z}(\kappa), 1 \leq \kappa < k\}.$$

In addition to generally replacing t by k, the variables $\mathbf{B}(t-L)$ are replaced by $\mathbf{B}(k-L-1)$, and $\mathbf{I}(t-L)$ are replaced by $\mathbf{I}(k-L-1)$. With these changes the proof for the periodic-review model proceeds precisely as the above for the continuous-review model.

Derivation of Equations 22-23

To prove (22), because $b_1 \geq b_2$, it is optimal to satisfy demand for product 1 first and then use the remaining parts to serve product 2. Thus the second-stage recourse problem (15) has the optimal solution:

$$z_1 = D_1 \wedge y = D_1 - (D_1 - y)^+ \quad \text{and} \quad z_2 = D_2 \wedge (y - D_1)^+ = D_2 - (D_2 - (y - D_1)^+)^+.$$

Inserting the above into (14),

$$\begin{aligned} C_s(y) =& b_1 \mathsf{E}[D_1] + b_2 \mathsf{E}[D_2] + hy - (b_1 + h)\mathsf{E}[D_1 - (D_1 - y)^+] \\ & - (b_2 + h)\mathsf{E}[D_2 - (D_2 - (y - D_1)^+)^+] \\ =& b_1 \mathsf{E}[(D_1 - y)^+] + b_2 \mathsf{E}[(D_2 - (y - D_1)^+)^+] \\ & + h\mathsf{E}[y - D_1 - D_2 + (D_1 - y)^+ + (D_2 - (y - D_1)^+)^+], \end{aligned}$$

and (22) follows because

$$y - D_1 - D_2 + (D_1 - y)^+ + (D_2 - (y - D_1)^+)^+ = (y - D_1 - D_2)^+.$$

To prove (23), because $b_1 \geq b_2$, the recourse problem (17) has the optimal solution:

$$\begin{aligned} z_1 &= (D_1 + \alpha_1) \wedge y, \\ z_2 &= (D_2 + \alpha_2) \wedge (y - D_1 - \alpha_1)^+. \end{aligned}$$

Inserting the above into (18),

$$\begin{aligned} \underline{C}_s(y, \alpha) =& b_1 \mathsf{E}[D_1 + \alpha_1] + b_2 \mathsf{E}[D_2 + \alpha_2] + hy \\ & - (b_1 + h)\mathsf{E}[(D_1 + \alpha_1) \wedge y] - (b_2 + h)\mathsf{E}[(D_2 + \alpha_2) \wedge (y - D_1 - \alpha_1)^+]. \end{aligned} \tag{EC-4}$$

We can assume that $y \geq \alpha_1 + \alpha_2$ without loss of optimality. If $y < \alpha_1 + \alpha_2$, then increasing y by $\Delta y = \alpha 1 + \alpha 2 - y$ in (EC-4) reduces $\underline{C}_s(y, \alpha)$ by $b_1 \Delta y$ on sample paths where $D_1 + \alpha_1 \geq y + \Delta y$, by $b_2 \Delta y$ on paths where $y > D_1 + \alpha_1$, and by some value in between on all other paths. Thus we can replace y in (EC-4) with $y + \alpha_1 + \alpha_2$ (where $y \geq 0$) and transform the equation into

$$\begin{aligned} \underline{C}_s(y, \alpha) =& b_1 \mathsf{E}[D_1] + b_2 \mathsf{E}[D_2 + \alpha_2] + h(y + \alpha_2) \\ & - (b_1 + h)\mathsf{E}[D_1 \wedge (y + \alpha_2)] - (b_2 + h)\mathsf{E}[(D_2 + \alpha_2) \wedge (y - D_1 + \alpha_2)^+]. \end{aligned} \tag{EC-5}$$

For any given y, $\underline{C}_s(y, \alpha)$ always improves with a higher α_2 because an increase of α_2 does not make any difference on sample paths where $D_1 \leq y + \alpha_2$, but reduces $\underline{C}_s(y, \alpha)$ on other paths. Therefore to reach the minimum of $\underline{C}_s(y, \alpha)$, $\alpha_2 \to \infty$, in which case

$$\begin{aligned} \mathsf{E}[D_1 \wedge (y + \alpha_2)] &\to \mathsf{E}[D_1], \\ \mathsf{E}[(D_2 + \alpha_2) \wedge (y - D_1 + \alpha_2)^+] - \alpha_2 &\to \mathsf{E}[(D_1 + D_2) \wedge y] - \mathsf{E}[D_1]. \end{aligned}$$

Inserting the above into (EC-5),

$$\lim_{\alpha_2 \to \infty} \underline{C}_s(y, \alpha) = b_2 \mathsf{E}[D_1 + D_2] + hy - (b_2 + h)\mathsf{E}[(D_1 + D_2) \wedge y],$$

which proves (23).

Proof of Lemma 3.1

Inserting $z_1(\mathbf{y},\boldsymbol{\alpha})$, $z_2(\mathbf{y},\boldsymbol{\alpha})$ into (25),

$$\begin{aligned}\underline{C}_s^* = \min_{\mathbf{y},\boldsymbol{\alpha}\geq 0}\Big\{&\sum_{j=0}^{2} h_j y_j + \sum_{i=1}^{2} b_i(\alpha_i + \mathsf{E}[D_i]) - c_1\mathsf{E}\big[(D_1+\alpha_1)\wedge y_0 \wedge y_1\big] \\ &- c_2\mathsf{E}\big[(D_2+\alpha_2)\wedge y_2 \wedge (y_0 - (D_1+\alpha_1)\wedge y_0\wedge y_1)\big]\Big\}. \qquad \text{(EC-6)}\end{aligned}$$

If $y_1 > y_0$, then $(D_1+\alpha_1)\wedge y_0\wedge y_1 = (D_1+\alpha_1)\wedge y_0$, and reducing y_1 to y_0 decreases $\underline{C}_s^*$ by $h_1(y_1 - y_0)$. Therefore, we can assume $y_1 \leq y_0$ and transform (EC-6) to

$$\begin{aligned}\underline{C}_s^* = \min_{\mathbf{y},\boldsymbol{\alpha}\geq 0}\Big\{&\sum_{j=0}^{2} h_j y_j + \sum_{i=1}^{2} b_i(\alpha_i + \mathsf{E}[D_i]) - c_1\mathsf{E}\big[(D_1+\alpha_1)\wedge y_1\big] \\ &- c_2\mathsf{E}\big[(D_2+\alpha_2)\wedge y_2 \wedge (y_0 - (D_1+\alpha_1)\wedge y_1)\big]\Big\}. \qquad \text{(EC-7)}\end{aligned}$$

If $y_1 < \alpha_1$, then $(D_1+\alpha_1)\wedge y_1 = y_1$ and the cost to be minimized in (EC-7) becomes

$$\begin{aligned}&\sum_{j=0}^{2} h_j y_j + \sum_{i=1}^{2} b_i(\alpha_i + \mathsf{E}[D_i]) - c_1 y_1 - c_2\mathsf{E}\big[(D_2+\alpha_2)\wedge y_2\wedge(y_0 - y_1)\big] \\ &= h_0(y_0 - y_1) + h_2 y_2 + \sum_{i=1}^{2} b_i\mathsf{E}[D_i] + b_1(\alpha_1 - y_1) + b_2\alpha_2 - c_2\mathsf{E}\big[(D_2+\alpha_2)\wedge y_2\wedge(y_0-y_1)\big],\end{aligned}$$

which can be reduced by raising y_1 to α_1 and increasing y_0 to keep $y_0 - y_1$ constant. Without loss of optimality, we can thus assume $y_0 \geq y_1 \geq \alpha_1$. Let $y_0' = y_0 - \alpha_1 \geq 0$, $y_1' = y_1 - \alpha_1 \geq 0$, and $y_2' = y_2$, so that

$$(D_1+\alpha_1)\wedge y_1 = \alpha_1 + D_1\wedge y_1', \quad y_0 - (D_1+\alpha_1)\wedge y_1 = y_0' - D_1\wedge y_1',$$

and (EC-7) becomes

$$\begin{aligned}\underline{C}_s^* = \min_{\mathbf{y}',\alpha_2\geq 0}\Big\{&\sum_{j=0}^{2} h_j y_j' + \sum_{i=1}^{2} b_i\mathsf{E}[D_i] + b_2\alpha_2 - c_1\mathsf{E}\big[D_1\wedge y_1'\big] \\ &- c_2\mathsf{E}\big[(D_2+\alpha_2)\wedge y_2'\wedge(y_0' - D_1\wedge y_1')\big]\Big\}. \qquad \text{(EC-8)}\end{aligned}$$

If $y_2' > y_0'$, then $y_2'\wedge(y_0' - D_1\wedge y_1') = y_0' - D_1\wedge y_1'$, so letting $y_2' = y_0'$ reduces the above cost by $h_2(y_2' - y_0')$. This leads to $y_2' \leq y_0'$. Moreover, if $y_2' < \alpha_2$, then

$$(D_2+\alpha_2)\wedge y_2'\wedge(y_0' - D_1\wedge y_1') = y_2' - (y_2' - y_0' + D_1\wedge y_1')^+.$$

By taking out y_2' from inside the last expectation in (EC-8), we transform $\underline{C}_s^*$ into

$$h_0(y_0' - y_2') + h_1 y_1' + \sum_{i=1}^{2} b_i\mathsf{E}[D_i] + b_2(\alpha_2 - y_2') - c_1\mathsf{E}\big[D_1\wedge y_1'\big] + c_2\mathsf{E}\Big[\big(y_2' - y_0' + D_1\wedge y_1'\big)^+\Big],$$

which can be reduced by raising y_2' to α_2 and y_0' by $\alpha_2 - y_2'$. Therefore, we can assume that $y_0' \ge y_2' \ge \alpha_2$. Let $y_0'' = y_0' - \alpha_2 \ge 0$, $y_1'' = y_1'$, and $y_2'' = y_2' - \alpha_2 \ge 0$, so

$$(D_2 + \alpha_2) \wedge y_2' \wedge (y_0' - D_1 \wedge y_1') = \alpha_2 + D_2 \wedge y_2'' \wedge (y_0'' - D_1 \wedge y_1''),$$

and thus (EC-8) becomes

$$\underline{C}_s^* = \min_{\mathbf{y}'' \ge 0} \Big\{ \sum_{j=0}^{2} h_j y_j'' + \sum_{i=1}^{2} b_i \mathsf{E}[D_i] - c_1 \mathsf{E}\big[D_1 \wedge y_1''\big] - c_2 \mathsf{E}\big[D_2 \wedge y_2'' \wedge (y_0'' - D_1 \wedge y_1'')\big] \Big\}. \quad \text{(EC-9)}$$

Proof of the Statement that "...finding boundary solutions is equivalent to solving standard newsvendor models."

We show that the problem of optimizing $C_s(\mathbf{y})$ or $\underline{C}_s(\mathbf{y})$ at a boundary can be transformed into the following standard form of the newsvendor model:

$$\min_{k \ge 0} \{hk - c\mathsf{E}[D \wedge k]\}.$$

If $y_1 + y_2 = y_0$, then

$$\begin{aligned} \underline{C}_s^* = C_s^* &= \sum_{i=1}^{2} b_i \mathsf{E}[D_i] + \min_{y \ge 0} \big\{ h_0(y_1 + y_2) + h_1 y_1 + h_2 y_2 \\ &\quad - c_1 \mathsf{E}[y_1 \wedge D_1] - c_2 \mathsf{E}[D_2 \wedge y_2 \wedge (y_1 + y_2 - y_1 \wedge D_1)] \big\} \\ &= \sum_{i=1}^{2} b_i \mathsf{E}[D_i] + \min_{y_1 \ge 0} \big\{ (h_0 + h_1) y_1 - c_1 \mathsf{E}[y_1 \wedge D_1] \big\} + \min_{y_2 \ge 0} \big\{ (h_0 + h_2) y_2 - c_2 \mathsf{E}[y_2 \wedge D_2] \big\}. \end{aligned}$$

If $y_1 = 0$, then

$$\begin{aligned} \underline{C}_s^* = C_s^* &= \sum_{i=1}^{2} b_i \mathsf{E}[D_i] + \min_{y_0, y_2 \ge 0} \big\{ h_0 y_0 + h_2 y_2 - c_2 \mathsf{E}[D_2 \wedge y_2 \wedge y_0] \big\} \\ &= \sum_{i=1}^{2} b_i \mathsf{E}[D_i] + \min_{y \ge 0} \big\{ (h_0 + h_2) y - c_2 \mathsf{E}[D_2 \wedge y] \big\} \text{ since } y_0 \ne y_2 \text{ is not optimal.} \end{aligned}$$

If $y_2 = 0$ and $y_1 \le y_0$, then $\mathsf{E}[D_2 \wedge y_2 \wedge (y_0 - y_1 \wedge D_1)] = 0$, so

$$\begin{aligned} C_s^* &= \sum_{i=1}^{2} b_i \mathsf{E}[D_i] + \min_{y_0 \ge y_1} \big\{ h_0 y_0 + h_1 y_1 - c_1 \mathsf{E}[y_1 \wedge D_1] \big\} \\ &= \sum_{i=1}^{2} b_i \mathsf{E}[D_i] + \min_{y \ge 0} \big\{ (h_0 + h_1) y - c_1 \mathsf{E}[y \wedge D_1] \big\}. \end{aligned}$$

If $y_2 = 0$ and $y_0 \le y_1$ (which is a possible case of (28)), then

$$D_2 \wedge y_2 \wedge (y_0 - D_1 \wedge y_1) = 0 \wedge (y_0 - D_1 \wedge y_1) = (D_1 \wedge y_0) - (D_1 \wedge y_1),$$

and therefore,

$$C_s(y_0, y_1, 0) = \sum_{i=1}^{2} b_i \mathsf{E}[D_i] + h_1 y_1 - (c_1 - c_2) \mathsf{E}[y_1 \wedge D_1] + h_0 y_0 - c_2 \mathsf{E}[D_1 \wedge y_0].$$

Let

$$\tilde{C}^* \equiv \sum_{i=1}^{2} b_i \mathsf{E}[D_i] + \min_{y_1 \geq 0}\{h_1 y_1 - (c_1 - c_2)\mathsf{E}[y_1 \wedge D_1]\} + \min_{y_0 \geq 0}\{h_0 y_0 - c_2 \mathsf{E}[D_1 \wedge y_0]\}.$$

Then $\underline{C}_s^* = C_s^* \wedge \tilde{C}_s^*$.

Proof of Theorem 3.2

We first consider a regular solution for (30). When $y_0 < y_1 + y_2$,

$$\begin{aligned} C_s(y_0, y_1, y_2) - C_s(y_0, y_1 - 1, y_2) &= h_1 - \bar{F}_1(y_1 - 1)[c_1 - c_2\bar{F}_2(y_0 - y_1)] = s_1(y_1|y_0), \\ C_s(y_0, y_1, y_2) - C_s(y_0, y_1, y_2 - 1) &= h_2 - c_2 F_1(y_0 - y_2)\bar{F}_2(y_2 - 1) = s_2(y_2|y_0). \end{aligned} \tag{EC-10}$$

With our convention that $F_i(k) = 0$ for $k < 0$, $s_2(y_2|y_0) = h_2$ if $y_2 > y_0$. By the definition of regular solution,

$$s_1(y_1^*|y_0^*) \leq 0 \;\text{ and }\; s_2(y_2^*|y_0^*) \leq 0.$$

Since, as noted previously, $s_i(y_i|y_0)$ $(i = 1, 2)$ weakly increases in y_i, by the definition of Y_i in (34),

$$s_i(Y_i(y_0^*)|y_0^*) \leq 0 < s_i(Y_i(y_0^*) + 1|y_0^*) \qquad i = 1, 2.$$

Because $0 < y_1^*$ and $0 < y_2^*$ hold for a regular solution, it is necessary that

$$s_1(1|y_0^*) \leq 0 \;\text{ and }\; s_2(1|y_0^*) \leq 0,$$

which means

$$h_1 - \bar{F}_1(0)[c_1 - c_2\bar{F}_2(y_0^* - 1)] \leq 0 \;\text{ and }\; h_2 - c_2 F_1(y_0^* - 1)\bar{F}_2(0) \leq 0,$$

which is true only if $Y_0^{min} \leq y_0^*$. For $Y_0^{min} < \infty$, it requires

$$h_i < c_i \bar{F}_i(0), \qquad i = 1, 2.$$

We now derive Y_0^{max}. From (EC-10), for $i = 1, 2$,

$$\text{if }\; c_i \bar{F}_i(y - 1) < h_i \;\text{ then }\; s_i(y|y_0) > 0.$$

Since $\bar{F}_i(y - 1)$ decreases in y and $s_i(Y_i(y_0^*)|y_0) \leq 0$ $(i = 1, 2)$,

$$Y_i(y_0^*) \leq \min\left\{k : \bar{F}_i(k - 1) < \frac{h_i}{c_i}\right\}, \quad i = 1, 2,$$

and because $y_0^* < y_1^* + y_2^* \leq Y_1(y_0^*) + Y_2(y_0^*)$, $y_0^* \leq Y_0^{max}$.

It is possible that $Y_1(y_0^*) > y_0$, i.e., $C_s(y_0^*, y_1, y_2^*)$ always decreases in y_1 within $[0, y_0]$. In this case, C_s is maximized at $y_1^* = y_0^*$, so in general, the optimal solution of (30) is $(y_0^*, y_0^* \wedge Y_1(y_0^*), Y_2(y_0^*))$, which can be obtained by a one-dimensional search of y_0^*.

The same proof goes through for optimizing $\underline{C}_s(\mathbf{y})$ except that y_1^* is not bounded by y_0^*, so the optimal solution takes the form $(y_0^*, Y_1(y_0^*), Y_2(y_0^*))$.

Derivation of Equation 35

For any nonnegative integer random variable D, let Y be a strictly positive integer, Y' be a nonnegative integer, and $Y' < Y$, then

$$\mathsf{E}[D \wedge Y - D \wedge Y'] = \sum_{k=Y'}^{Y-1} \mathsf{Pr}(D > k),$$

and as a special case, when $Y' = 0$,

$$\mathsf{E}[D \wedge Y] = \sum_{k=0}^{Y-1} \mathsf{Pr}(D > k).$$

In $\underline{C}_s(\mathbf{y})$ and $C_s(\mathbf{y})$, apply the above to $z_1^*(\mathbf{y})$,

$$\mathsf{E}[z_1^*(\mathbf{y})] = \mathsf{E}[D_1 \wedge y_1] = \sum_{k=0}^{y_1-1} \bar{F}_1(k).$$

Deriving $z_2^*(\mathbf{y})$ is more involved. Given that $y_0 \le y_1 + y_2$, if $y_1 \le y_0$, then

$$\begin{aligned}\mathsf{E}[z_2^*(\mathbf{y})] &= \mathsf{E}[D_2 \wedge y_2 \wedge (y_0 - D_1 \wedge y_1)] \\ &= \mathsf{E}[D_2 \wedge (y_0 - y_1)] + \sum_{k=0}^{y_1-1} f_1(k)\Big[\mathsf{E}[D_2 \wedge y_2 \wedge (y_0 - k)] - \mathsf{E}[D_2 \wedge (y_0 - y_1)]\Big],\end{aligned}$$

where $f_i(k) = \mathsf{Pr}(D_i = k)$ $(i = 1, 2)$. The first part of equation (35) follows because

$$\mathsf{E}[D_2 \wedge (y_0 - y_1)] = \sum_{k=0}^{y_0-y_1-1} \bar{F}_2(k)$$

and

$$\begin{aligned}&\sum_{k=0}^{y_1-1} f_1(k)\Big[\mathsf{E}[D_2 \wedge y_2 \wedge (y_0 - k)] - \mathsf{E}[D_2 \wedge (y_0 - y_1)]\Big] \\ =& F_1(y_0 - y_2 - 1)\Big[\mathsf{E}[D_2 \wedge y_2 - D_2 \wedge (y_0 - y_1)]\Big] + \sum_{k=y_0-y_2}^{y_1-1} f_1(k)\Big[\mathsf{E}[D_2 \wedge (y_0 - k) - D_2 \wedge (y_0 - y_1)]\Big] \\ =& F_1(y_0 - y_2 - 1) \sum_{k=y_0-y_1}^{y_2-1} \bar{F}_2(k) + \sum_{k=y_0-y_2}^{y_1-1} f_1(k) \sum_{k'=y_0-y_1}^{y_0-k-1} \bar{F}_2(k') \\ =& F_1(y_0 - y_2 - 1) \sum_{k=y_0-y_1}^{y_2-1} \bar{F}_2(k) + \sum_{k=y_0-y_1}^{y_2-1} \bar{F}_2(k) \sum_{k'=y_0-y_2}^{y_0-k-1} f_1(k') \\ =& \sum_{k=y_0-y_1}^{y_2-1} F_1(y_0 - k - 1)\bar{F}_2(k).\end{aligned}$$

If $y_1 > y_0$ (which only applies to $\underline{C}_s(\mathbf{y})$),

$$\begin{aligned}\mathsf{E}[z_2^*(\mathbf{y})] &= \mathsf{E}[D_2 \wedge y_2 \wedge (y_0 - D_1 \wedge y_1)] \\ &= F_1(y_0 - y_2 - 1)\mathsf{E}[y_2 \wedge D_2] + \sum_{k=y_0-y_2}^{y_0-1} f_1(k)\mathsf{E}[D_2 \wedge (y_0 - k)] + \mathsf{E}[(y_0 - D_1 \wedge y_1)^-] \\ &= F_1(y_0 - y_2 - 1) \sum_{k=0}^{y_2-1} \bar{F}_2(k) + \sum_{k=y_0-y_2}^{y_0-1} f_1(k) \sum_{k'=0}^{y_0-k-1} \bar{F}_2(k') - \mathsf{E}[D_1 \wedge y_1 - D_1 \wedge y_0] \\ &= \sum_{k=0}^{y_2-1} F_1(y_0 - k - 1)\bar{F}_2(k) - \sum_{k=y_0}^{y_1-1} \bar{F}_1(k),\end{aligned}$$

which leads to the second part of (35).

Proof of Lemma 3.3

When $y_0 > y_1 + y_2$, then by (36) and (37)

$$I_0^p(t) = y_0 - y_1 - y_2 + I_1^p(t) + I_2^p(t) > 0 \quad \text{for all} \quad t \geq 0. \tag{EC-11}$$

Combining (37) and (38) we obtain

$$B_i^p(t) = D_i(t) - y_i + I_i^p(t), \quad i = 1, 2. \tag{EC-12}$$

From (39) and (EC-11) we have

$$I_i^p(t)B_i^p(t) = 0, \quad i = 1, 2. \tag{EC-13}$$

Equation (40) is immediate from (EC-12) and (EC-13), along with the condition that $I_i^p(t), B_i^p(t) \geq 0$ $(i = 1, 2)$.

We now prove (41). By canceling out $B_i^p(t - L) - Z_i^p(t)$ $(i = 1, 2)$ in (36)-(38),

$$\begin{aligned} D_1(t) + D_2(t) - y_0 &= B_1^p(t) + B_2^p(t) - I_0^p(t), \\ D_i(t) - y_i &= B_i^p(t) - I_i^p(t), \qquad i = 1, 2. \end{aligned}$$

Because $I_j^p(t) \geq 0$ $(j = 0, 1, 2)$,

$$B_1^p(t) + B_2^p(t) \geq (D_1(t) + D_2(t) - y_0)^+ \vee (D_1(t) - y_1)^+ \vee (D_2(t) - y_2)^+,$$

so we can prove (41) by showing that

$$B_1^p(t) + B_2^p(t) \leq (D_1(t) + D_2(t) - y_0)^+ \vee (D_1(t) - y_1)^+ \vee (D_2(t) - y_2)^+. \tag{EC-14}$$

When $y_0 \leq y_1 + y_2$, equations (36) and (37) imply that

$$I_0^p(t) = I_1^p(t) + I_2^p(t) + y_0 - y_1 - y_2 \leq I_1^p(t) + I_2^p(t).$$

Hence $I_0^p(t) = 0$ if $I_1^p(t) = I_2^p(t) = 0$. In order for the defining property of a myopic policy

$$[I_i^p(t) \wedge I_0^p(t)]B_i^p(t) = 0, \quad i = 1, 2,$$

to hold, at least one of the following must be true:

- $B_1^p(t) = B_2^p(t) = 0$, in which case (EC-14) is obviously satisfied.
- $I_i^p(t) = 0$ and $B_{-i}^p(t) = 0$ $(i = 1, 2)$, in which case (EC-14) is true because

$$B_1^p(t) + B_2^p(t) = B_i^p(t) = D_i(t) - y_i.$$

- $I_0^p(t) = 0$, in which case (EC-14) is also true because

$$B_1^p(t) + B_2^p(t) = D_1(t) + D_2(t) - y_0.$$

To prove the second equality in (41),

$$\begin{aligned} D_1 - z_1^*(\mathbf{y}) + D_2 - z_2^*(\mathbf{y}) &= D_1 - D_1 \wedge y_1 + D_2 - D_2 \wedge y_2 \wedge (y_0 - D_1 \wedge y_1) \\ &= (D_1 - y_1)^+ + (D_2 - y_2 \wedge (y_0 - D_1 \wedge y_1))^+. \end{aligned} \tag{EC-15}$$

- If $D_2 < y_2 \wedge (y_0 - D_1 \wedge y_1)$, then $(D_2 - y_2)^+ = 0$, and (EC-15) shows that

$$D_1 - z_1^*(\mathbf{y}) + D_2 - z_2^*(\mathbf{y}) = (D_1 - y_1)^+,$$

and

$$(D_1 + D_2 - y_0)^+ \leq (D_1 + y_0 - D_1 \wedge y_1 - y_0)^+ = (D_1 - y_1)^+.$$

- If $D_2 \geq y_2 \wedge (y_0 - D_1 \wedge y_1)$ and $D_1 < y_1$, then $(D_1 - y_1)^+ = 0$, and by (EC-15),

$$D_1 - z_1^*(\mathbf{y}) + D_2 - z_2^*(\mathbf{y}) = (D_2 - y_2 \wedge (y_0 - D_1))^+ = (D_1 + D_2 - y_0)^+ \vee (D_2 - y_2)^+.$$

- If $D_2 \geq y_2 \wedge (y_0 - D_1 \wedge y_1)$ and $D_1 \geq y_1$, then by (EC-15),

$$D_1 - z_1^*(\mathbf{y}) + D_2 - z_2^*(\mathbf{y}) \geq (D_1 - y_1)^+, .$$

Because $y_0 \leq y_1 + y_2$, $D_2 \geq y_0 - y_1$. Thus $D_1 + D_2 - y_0 \geq 0$ and

$$D_1 - z_1^*(\mathbf{y}) + D_2 - z_2^*(\mathbf{y}) = D_1 - y_1 + D_2 - y_2 \wedge (y_0 - y_1) = (D_1 + D_2 - y_0)^+ \geq (D_2 - y_2)^+.$$

This proves that the equality holds in all cases.

Proof of Theorem 3.4

The following proof is based on the continuous review model but can be directly extended to the periodic review model. From (36)-(38),

$$I_0^p(t) = y_0 + B_1^p(t) + B_2^p(t) - D_1(t) - D_2(t), \; I_i^p(t) = y_i + B_i^p(t) - D_i(t), \; (i = 1, 2),$$

so in (8),

$$\sum_{i=1}^{2} b_i \mathsf{E}[B_i^p(t)] + \sum_{j=0}^{2} h_j \mathsf{E}[I_j^p(t)] = \sum_{j=0}^{2} h_j y_j - \sum_{i=1}^{2} (h_0 + h_i) \mathsf{E}[D_i(t)] + \sum_{i=1}^{2} c_i \mathsf{E}[B_i^p(t)]. \tag{EC-16}$$

In (30),

$$\sum_{j=0}^{2} h_j y_j + \sum_{i=1}^{2} b_i \mathsf{E}[D_i] - \sum_{i=1}^{2} c_i \mathsf{E}[z_i^*(\mathbf{y})] = \sum_{j=0}^{2} h_j y_j - \sum_{i=1}^{2} (h_0 + h_i) \mathsf{E}[D_i] + \sum_{i=1}^{2} c_i \mathsf{E}[D_i - z_i^*(\mathbf{y})]. \tag{EC-17}$$

The right hand sides of (EC-16) and (EC-17) are equal when $c_1 = c_2$ and $D_i(t) = D_i$ because by Lemma 3.3, $B_1^p(t) + B_2^p(t) = D_1 + D_2 - z_1^*(\mathbf{y}) - z_2^*(\mathbf{y})$. This completes the proof of (44).

To prove (45), note that, by Lemma 3.3 it is sufficient to show that

$$B_1^p(t) \geq B_1^{PBC}(t) \tag{EC-18}$$

for any $p \in \mathcal{P}_m$ and $t \geq 0$. Suppose that (EC-18) does not hold, and let τ denote the first time that (EC-18) is violated. Then $\Delta\mathcal{Z}_1^p(\tau) > \Delta\mathcal{Z}_1^{PBC}(\tau) + \delta^-$, where $\delta^- \equiv B_1^p(\tau^-) - B_1^{PBC}(\tau^-) \geq 0$. By (42) we have

$$\Delta\mathcal{Z}_1^p(\tau) > \min\{B_1^{PBC}(\tau^-) + \Delta\mathcal{D}_1(\tau), \; I_0^{PBC}(\tau^-) + \Delta\mathcal{R}_0(\tau), \; I_1^{PBC}(\tau^-) + \Delta\mathcal{R}_1(\tau)\} + \delta^-.$$

There are three possible cases:

1. $\Delta\mathcal{Z}_1^p(\tau) > B_1^{PBC}(\tau^-) + \Delta\mathcal{D}_1(\tau) + \delta^- = B_1^p(\tau^-) + \Delta\mathcal{D}_1(\tau)$.
   Using (6) with $t = \tau$ it is clear that in this case $B_1^p(\tau) < 0$, so p is not feasible.

2. $\Delta\mathcal{Z}_1^p(\tau) > I_0^{PBC}(\tau^-) + \Delta\mathcal{R}_0(\tau) + \delta^-$.
   Note that, by Lemma 3.3, $I_0^p(t) = I_0^{PBC}(t)$ for all $t \geq 0$. Also, $\Delta\mathcal{Z}_2(t) \geq 0$ for all $t \geq 0$. Thus, using (7) with $j = 0$ and $t = \tau$, we obtain

$$I_0^p(\tau) < I_0^p(\tau^-) - I_0^{PBC}(\tau^-) - \Delta\mathcal{Z}_2(\tau) - \delta^- \leq 0,$$

   so p is not feasible.

3. $\Delta\mathcal{Z}_1^p(\tau) > I_1^{PBC}(\tau^-) + \Delta\mathcal{R}_1(\tau) + \delta^-$.
   We have $I_1^p(t) - I_1^{PBC}(t) = B_1^p(t) - B_1^{PBC}(t)$ for all $t \geq 0$, so that $I_1^p(t) - I_1^{PBC}(t) = -\delta^-$. Using (7) with $j = 1$ and $t = \tau$ we obtain

$$I_1^p(\tau) < I_1^p(\tau^-) - I_1^{PBC}(\tau^-) - \delta^- = -2\delta^-,$$

   so, again, p is not feasible.

Thus (EC-18) always holds.

Proof of Theorem 3.5

When $c_1 = c_2$, it is immediate from (44) that $C_s^*(\mathbf{y}^*) = C^{PBC}(\mathbf{y}^*)$, so we only need to show that $C_s^*(\mathbf{y}^*) = \underline{C}_s^*$. This is proved by showing that if $\mathbf{y}^{**}$ is the optimal solution of (31), then $y_1^{**} \leq y_0^{**}$, so $\mathbf{y}^{**}$ is a feasible solution of (30).

Suppose $y_0^{**} < y_1^{**}$, then (31) applies. For the discrete demand formulation,

$$\underline{C}_s(y_0^{**}, y_1^{**} - 1, y_2^{**}) - \underline{C}_s(y_0^{**}, y_1^{**}, y_2^{**}) = -h_1 + (c_1 - c_2)\bar{F}_1(y_1^{**} - 1).$$

Because y_1^{**} is optimal, the above has to be nonnegative. When $c_1 = c_2$, this means $-h_1 \geq 0$, which does not hold, so $y_0^{**} < y_1^{**}$ cannot be true.

The same argument applies to the continuous demand formulation if we use $\partial\underline{C}_s/\partial y_1$, evaluated at $\mathbf{y}^{**}$, to replace $\underline{C}_s(y_0^{**}, y_1^{**} - 1, y_2^{**}) - \underline{C}_s(y_0^{**}, y_1^{**}, y_2^{**})$ in the above.

When $y_1^* + y_2^* = y_0^*$, $y_0^* - D_1 \wedge y_1^* \geq y_2^*$, so in (30),

$$\begin{aligned} C_s(y_0^*, y_1^*, y_2^*) =& (h_0 + h_1)y_1^* + (h_0 + h_2)y_2^* + \sum_{i=1}^{2} b_i \mathsf{E}[D_i] - c_1\mathsf{E}[D_1 \wedge y_1^*] - c_2\mathsf{E}[D_2 \wedge y_2^*] \\ =& \sum_{i=1}^{2}(h_i + h_0)\mathsf{E}[(y_i^* - D_i)^+] + \sum_{i=1}^{2} b_i\mathsf{E}[(D_i - y_i^*)^+], \end{aligned} \tag{EC-19}$$

where the second equality makes use of $D_i \wedge y_i^* = y_i^* - (D_i - y_i^*)^+ = D_i - (D_i - y_i^*)^+$ $(i = 1, 2)$.

In the inventory system, equations (41), (EC-12), along with conditions that both inventory and backlog levels are nonnegative, imply that

$$\begin{aligned} B_i^{PBC}(t) &= (D_i(t) - y_i^*)^+, \quad I_i^{PBC}(t) = (y_i^* - D_i(t))^+, \qquad (i = 1, 2) \\ \text{and } \; I_0^{PBC}(t) &= (y_1^* - D_1(t))^+ + (y_2^* - D_2^*(t))^+, \end{aligned}$$

which together with (8) and (EC-19), imply that

$$C_s^*(\mathbf{y}^*) = C^{PBC}(\mathbf{y}^*).$$

It remains to be shown that $C_s^*(\mathbf{y}^*) = \underline{C}_s^*$. Following the same argument as above, we prove that there exists at least one optimal solution $\mathbf{y}^{**}$ for (31) such that $y_1^{**} \leq y_0^{**}$, so $\mathbf{y}^{**}$ is a feasible solution of (30).

We first consider the discrete demand formulation. Because y_0^* and y_1^* are optimal for (30),

$$C_s(y_0^* - 1, y_1^*, y_2^*) - C_s(y_0^*, y_1^*, y_2^*) = -h_0 + c_2 \mathsf{Pr}\{D_1 \geq y_1^*, D_2 \geq y_2^*\} \geq 0$$
$$C_s(y_0^*, y_1^* + 1, y_2^*) - C_s(y_0^*, y_1^*, y_2^*) = h_1 - c_1 \bar{F}_1(y_1^*) + c_2 \mathsf{Pr}\{D_1 > y_1^*, D_2 \geq y_2^*\} \geq 0,$$

which implies that

$$\bar{F}_1(y_1^*) \leq \frac{h_1}{c_1 - c_2} \quad \text{and} \quad \bar{F}_1(y_1^* - 1) \geq \frac{h_0}{c_2}. \tag{EC-20}$$

The optimal solution of (31) may not be unique. We choose $\mathbf{y}^{**}$ such that $y_1^{**} - y_0^{**}$ is the minimum of all optimal solutions, which ensures the following *strict* inequalities

$$\underline{C}_s(y_0^{**} + 1, y_1^{**}, y_2^{**}) - \underline{C}_s(y_0^{**}, y_1^{**}, y_2^{**}) > 0,$$
$$\underline{C}_s(y_0^{**}, y_1^{**} - 1, y_2^{**}) - \underline{C}_s(y_0^{**}, y_1^{**}, y_2^{**}) > 0.$$

Using (31) to expand the left-hand sides of the above,

$$h_0 - c_2 \mathsf{Pr}\{D_2 \wedge y_2^{**} > y_0^{**} - D_1 \geq 0\} - c_2 \bar{F}_1(y_0^{**}) > 0,$$
$$-h_1 + (c_1 - c_2)\bar{F}_1(y_1^{**} - 1) > 0,$$

which implies that

$$\bar{F}_1(y_0^{**}) < \frac{h_0}{c_2} \quad \text{and} \quad \bar{F}_1(y_1^{**} - 1) > \frac{h_1}{c_1 - c_2}. \tag{EC-21}$$

Following from (EC-20) and (EC-21),

$$y_1^* - 1 < y_0^{**} \text{ and } y_1^{**} - 1 < y_1^*.$$

Since y_0^{**}, y_1^{**}, and y_1^* are all integers, the above holds only if $y_0^{**} \geq y_1^{**}$.

A similar argument carries over to the continuous demand formulation. By using partial derivatives in place of differences of C_s and $\underline{C}_s$, we arrive at following analogous inequalities to (EC-20) and (EC-21),

$$\frac{h_0}{c_2} \leq \bar{F}_1(y_1^*) \leq \frac{h_1}{c_1 - c_2}, \; \bar{F}_1(y_0^{**}) \leq \frac{h_0}{c_2}, \text{ and } \bar{F}_1(y_1^{**}) \geq \frac{h_1}{c_1 - c_2},$$

so $y_1^{**} \leq y_1^* \leq y_0^{**}$.

II. Periodic-Review Formulation

In the following, we develop a periodic-review (discrete time) formulation that exactly parallels the continuous-review model presented in Section 2. With slight modification to definitions, all results in the paper carry over.

There are m products and n components and a_{ij} is the amount of component j $(1 \leq j \leq n)$ needed to assemble product i $(1 \leq i \leq m)$. Let $k \geq 1$ be the index of review periods. Let h_j denote the inventory holding cost per-period of component j $(1 \leq j \leq n)$ and b_i denote the backlog cost per-period of product i $(1 \leq i \leq m)$. At the beginning of period k, an order of $\mathbf{r}(k) \equiv \{r_1(k), ..., r_n(k)\}$ for components $1 \leq j \leq n$ is placed based on a replenishment policy γ. All components have a common, deterministic replenishment lead time L. Both product demands in this period, $d_i(k), 1 \leq i \leq m$, and component orders placed L periods earlier, $r_j(k - L), 1 \leq j \leq n$, arrive by the end of period k. For $k \geq 1$, let $\mathbf{d}(k) \equiv (d_1(k), \ldots, d_m(k))$. We assume that $\{\mathbf{d}(k), k \geq 1\}$ is an i.i.d. sequence. We also assume that $\mathsf{E}[d_i(1)] < \infty$ for all $1 \leq i \leq m$. Under an allocation policy p, an amount $z_i(k)$ of product i demands are served, giving rise to end of period backlog levels

$$B_i(k) = B_i(k-1) + d_i(k) - z_i(k), \qquad 1 \leq i \leq m,$$

and end of period on-hand inventory levels

$$I_j(k) = I_j(k-1) + r_j(k-L) - \sum_{i=1}^{m} a_{ij} z_i(k), \quad 1 \leq j \leq n.$$

As for initial conditions, $I_j(0), 1 \leq j \leq n$, are the inventory levels and $B_i(0), 1 \leq i \leq m$, are the backlog levels at the beginning of the first period, and $r_j(1-L), ..., r_j(0)$ are the initial orders in the pipeline that will arrive by the end of periods $1, ..., L$, respectively.

Adding up each equation from $k - L$ to k and canceling out redundant terms, we get the following analogous conditions to (1) and (2) of the continuous-review model:

$$B_i(k) = B_i(k-L-1) + D_i(k) - Z_i(k), \quad 1 \leq i \leq m, \tag{EC-22}$$

$$I_j(k) = I_j(k-L-1) + R_j(k-L) - \sum_{i=1}^{m} a_{ij} Z_i(k), \quad 1 \leq j \leq n, \tag{EC-23}$$

where $R_j(k) \equiv r_j(k-L) + ... + r_j(k)$ is the amount of component j ordered from period $k-L$ to k, and $D_i(k) \equiv d_i(k-L) + ... + d_i(k)$ and $Z_i(k) \equiv z_i(k-L) + ... + z_i(k)$ are respectively the amounts of product i demand arrived and served from period $k - L$ to period k. The quantity $D_i(k)$ is also referred to as the lead-time demand.

We seek to minimize the long-run average expected cost, which for the periodic-review model is defined as

$$C^{\gamma,p} \equiv \limsup_{K \to \infty} \frac{1}{K} \mathsf{E} \left[\sum_{k=1}^{K} \left\{ \sum_{i=1}^{m} b_i B_i(k) + \sum_{j=1}^{n} h_j I_j(k) \right\} \right].$$

Parallel to the continuous-review formulation, a feasible policy for the periodic-review model must satisfy conditions that

1. For $k \geq 1$, $B_i(k) \geq 0, 1 \leq i \leq m$. From (EC-22), this implies that

$$Z_i(k) \leq B_i(k-L-1) + D_i(k), \qquad 1 \leq i \leq m.$$

2. For $k \geq 1$, $I_j(k) \geq 0, 1 \leq j \leq n$. From (EC-23), this implies that

$$\sum_{i=1}^{m} a_{ij} Z_i(k) \leq I_j(k-L-1) + R_j(k-L), \qquad 1 \leq j \leq n.$$

3. For all $k \geq 1$, $\mathbf{r}(k)$ is chosen using only the information $\mathbf{I}(0), \mathbf{B}(0)$, $\{\mathbf{d}(l), 1 \leq l < k\}, \{\mathbf{r}(l), 1 - L \leq l < k\}$, and $\{\mathbf{z}(l), 1 \leq l < k\}$, while $\mathbf{z}(k)$ is chosen using only the information $\mathbf{I}(0), \mathbf{B}(0)$, $\{\mathbf{d}(l), 1 \leq l \leq k\}, \{\mathbf{r}(l), 1 - L \leq l \leq k\}$, and $\{\mathbf{z}(l), 1 \leq l < k\}$.

III. Solving the Continuous Stochastic Program for the W System

At the end of Section 3.2, we state that "...because the continuous formulation yields smooth objective functions, both $C_s(\mathbf{y})$ and $\underline{C}_s(\mathbf{y})$ can be reduced to convex functions of y_0. Consequently, we can find y_0^* by a one-dimensional bisection search instead of enumerating y_0 over $[Y_0^{min}, Y_0^{max}]$." We now provide more details about this statement.

Analogous to $s_i(y_i|y_0)$ $(i = 1, 2)$ in (EC-10) for the discrete case, let

$$S_1(y_1|y_0) = \frac{\partial C_s(\mathbf{y})}{\partial y_1} = h_1 - \bar{F}_1(y_1)[c_1 - c_2\bar{F}_2(y_0 - y_1)],$$
$$S_2(y_2|y_0) = \frac{\partial C_s(\mathbf{y})}{\partial y_2} = h_2 - c_2 F_1(y_0 - y_2)\bar{F}_2(y_2),$$

in the continuous model. To minimize $C_s(\mathbf{y})$ at $y_1 > 0$, $y_2 > 0$ and $y_0 < y_1 + y_2$, it is necessary that

$$S_i(y_i|y_0) = 0, \qquad i = 1, 2. \tag{EC-24}$$

Because $S_i(y_i|y_0)$ $(i = 1, 2)$ increase in y_i, the solution exists only if $S_i(0|y_0) < 0$, i.e.,

$$y_0 \geq Y_0^{min} \equiv \bar{F}_2^{-1}\Big(\frac{c_1 - h_1}{c_2} \wedge 1\Big) \vee F_1^{-1}\Big(\frac{h_2}{c_2}\Big).$$

When (EC-24) is true, $c_i\bar{F}_i(y_i) \geq h_i$ $(i = 1, 2)$, so $y_0 < y_1 + y_2$ makes it necessary that

$$y_0 < Y_0^{max} \equiv \bar{F}_1^{-1}\Big(\frac{h_1}{c_1}\Big) + \bar{F}_2^{-1}\Big(\frac{h_2}{c_2}\Big).$$

It is easy to verify that when $y_0 \in [Y_0^{min}, Y_0^{max}]$,

$$S_1(0|y_0) < 0 < S_1(\infty|y_0), \quad \text{and} \quad S_2(0|y_0) < 0 < S_2(y_0|y_0),$$

and because under the assumption that D_i have strictly positive densities, $S_i(y_i|y_0)$ strictly increase in y_i $(i = 1, 2)$, (EC-24) has a unique solution $Y_1(y_0) \in (0, \infty)$ and $Y_2(y_0) \in (0, y_0)$.

We first consider cases where

$$Y_0^{min} \geq \tilde{Y}_0 \equiv \bar{F}_1^{-1}\Big(\frac{h_1}{c_1 - c_2} \wedge 1\Big), \tag{EC-25}$$

implying that $S_1(Y_0^{min}|Y_0^{min}) \geq 0$. Because $S_1(y_0|y_0)$ increases in y_0, (EC-25) means that for all $y_0 \in [Y_0^{min}, Y_0^{max}]$, $S_1(y_1|y_0) \geq 0$ at $y_1 = y_0$, and thus $Y_1(y_0) \leq y_0$.

At $(y_0, Y_1(y_0), Y_2(y_0))$,

$$\begin{aligned}\frac{dC_s}{dy_0} &= h_0 - c_2\big[\mathsf{Pr}(D_1 \geq Y_1, D_2 \geq y_0 - Y_1) + \mathsf{Pr}(y_0 - Y_2 \leq D_1 < Y_1, D_1 + D_2 \geq y_0)\big] \\ &= h_0 - c_2\Big[\bar{F}_1(Y_1)\bar{F}_2(y_0 - Y_1) + \int_{y_0 - Y_2}^{Y_1} f_1(x)\bar{F}_2(y_0 - x)dx\Big],\end{aligned}$$

and $dC_s/dy_0 = 0$ is the first-order necessary condition for the regular solution. Applying implicit differentiation to $S_1(y_1|y_0) = 0$ and $S_2(y_2|y_0) = 0$,

$$\frac{dY_1}{dy_0} = \frac{c_2\bar{F}_1(Y_1)f_2(y_0 - Y_1)}{f_1(Y_1)[c_1 - c_2\bar{F}_2(y_0 - Y_1)] + c_2\bar{F}_1(Y_1)f_2(y_0 - Y_1)} \in [0, 1], \tag{EC-26}$$

$$\frac{dY_2}{dy_0} = \frac{f_1(y_0 - Y_2)\bar{F}_2(Y_2)}{F_1(y_0 - Y_2)f_2(Y_2) + f_1(y_0 - Y_2)\bar{F}_2(Y_2)} \in [0, 1]. \tag{EC-27}$$

It follows that dC_s/dy_0 monotonically increase in y_0 because

$$\begin{aligned}\frac{d^2C_s(y_0)}{dy_0^2} =& c_2\bar{F}_1(Y_1)f_2(y_0-Y_1)\Big(1-\frac{dY_1}{dy_0}\Big)+c_2f_1(y_1)\bar{F}_2(y_0-Y_1)\Big(1-\frac{dY_1}{dy_0}\Big)\\ &+c_2f_1(y_0-Y_2)\bar{F}_2(Y_2)\Big(1-\frac{dY_2}{dy_0}\Big)+c_2\int_{y_0-Y_2}^{Y_1}f_1(x)f_2(y_0-x)dx\\ \geq& 0 \qquad \text{because } \frac{dY_1}{dy_0}, \frac{dY_2}{dy_0}\leq 1.\end{aligned}$$

Therefore, $dC_s/dy_0 = 0$ can be solved by a simple bisection search. The root is the regular solution if it satisfies $y_0 < Y_1(y_0) + Y_2(y_0)$. Otherwise, the constraint and the first-order condition cannot be satisfied simultaneously, indicating the regular solution does not exist.

For cases where $Y_0^{min} < \tilde{Y}_0$, we search for the regular solution in $[Y_0^{min}, \tilde{Y}_0]$ and $[\tilde{Y}_0, Y_0^{max}]$ separately. The same procedure as the one above applies to the search in the second interval while a minor change is needed for the search in the first interval. For $y_0 \in [Y_0^{min}, \tilde{Y}_0]$, $S_1(y_1|y_0) < 0$ at $y_1 = y_0$, so $Y_1(y_0) > y_0$, and given y_0, $y_1 = y_0$ and $y_2 = Y_2(y_0)$ minimize $C_s(\mathbf{y})$ under the constraint $y_1 \leq y_0$. Hence

$$C_s(y_0) = \sum_{i=1}^{2} b_i\mathsf{E}[D_i] + (h_0+h_1)y_0 + h_2Y_2(y_0) - c_1\mathsf{E}[y_0\wedge D_1] - c_2\mathsf{E}[D_2\wedge Y_2(y_0)\wedge(y_0-y_0\wedge D_1)].$$

We prove that $C_s(y_0)$ is convex over $[Y_0^{min}, \tilde{Y}_0]$ so a simple bisection search also suffices. Because $\partial C_s/\partial y_2 = 0$ at $y_2 = Y_2(y_0)$,

$$\begin{aligned}\frac{dC_s}{dy_0} &= h_1+h_0-c_1\bar{F}_1(y_0)-c_2\mathsf{Pr}(y_0-Y_2(y_0)\leq D_1\leq y_0\leq D_1+D_2)\\ &= h_1+h_0-c_1\bar{F}_1(y_0)-c_2\int_{y_0-Y_2(y_0)}^{y_0}f_1(x)\bar{F}_2(y_0-x)dx,\\ \frac{d^2C_s(y_0)}{dy_0^2} &= (c_1-c_2)f_1(y_0)+c_2f_1(y_0-Y_2(y_0))\bar{F}_2(Y_2(y_0))\Big(1-\frac{dY_2}{dy_0}\Big)\\ &\quad+\int_{y_0-Y_2(y_0)}^{y_0}f_1(x)f_2(y_0-x)dx\\ &\geq 0 \qquad \text{because } \frac{dY_2}{dy_0}\leq 1 \text{ by (EC-26).}\end{aligned}$$

Note for $Y_0^{min} < y_0 \leq \tilde{Y}_0$, $y_1 = y_0$, $y_2 > 0$, so the constraint $y_0 < y_1 + y_2$ is always satisfied here.

The same approach applies to the optimization of $\underline{C}_s(\mathbf{y})$ except that there is no need to divide the search into two regions when $Y_0^{min} < \tilde{Y}_0$ because the constraint $y_1 \leq y_0$ no longer applies.